

National University of Science and Technology Politehnica Bucharest
Department of Mathematics & Informatics
Splaiul Independenței 313
060042 Bucharest, Romania

Hemivariational inequality systems and best proximity points

PhD student Monica STANCIU (CONSTANTIN)
E-mail address: `monica.stanciu2303@upb.ro`

Abstract of the PhD Thesis

Supervised by Prof. Dr. Ariana PITEA

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Abstract

General objective. The research area of this thesis is nonlinear analysis. On one hand, we study a variational-hemivariational type inequality system, with variable constraint sets that depend on the solution set of the system. As an application of the theoretical results, we consider a mathematical model which describes the contact between a deformable elastic body and a Winkler type foundation. On the other hand, we develop best proximity point results in various context for classes of mappings satisfying weak contractive type conditions. As consequences, we obtain applications in fixed point theory.

The conclusions drawn highlight several characteristics of the conducted study:

- The main concerns of this study are, on one hand, proving that the solution set of a problem with variational-hemivariational inequalities is nonempty and weakly compact, when the constraint sets are bounded. For the complementary case, using adequate coercivity conditions, it is showed that the solution set of this problem is nonempty. On the other hand, in the best proximity point study, the main goal is to enunciate some conditions in which the distance from a point to its image through a mapping, reaches its minimum possible value. Depending on the metric structure that is taken into consideration, additional conditions are imposed to ensure the existence, and in some cases, also the uniqueness of the solution for the considered problem.
- In this thesis there are used, on one hand, with regard to variational-hemivariational inequalities, qualitative properties of the solution for the problem of study, the minimax principle, fixed point techniques, and coercivity conditions. On the other hand, with regards to best proximity points, are considered classes of mappings that respect some contractive type properties and classes of Istrăţescu type contractions, whose existence is studied in the framework of metric spaces, b -metric spaces, and generalized metric spaces.

Methodology. The methodology used in this thesis draws on specific tools from both convex analysis and nonsmooth analysis. Concerning the systems with variational-hemivariational inequalities, several cases are considered, depending on the properties of the constraint sets. The system is decoupled by associating an auxiliary problem, followed by the use of a fixed point technique. The mathematical model considered to study the contact between a deformable body and a Winkler type foundation uses the primal-dual formulation, which leads to a coupled system of variational-hemivariational inequalities, that allows the simultaneous determination of the displacement field and stress tensor.

With regards to best proximity points, a constrained optimization problem is considered. Sets of proximal pairs are used, together with specific techniques applied for some adequate classes of mappings, to formulate results for the existence of such points. Implicit and explicit type contractions are formulated using classes of functions that satisfy adequate properties with respect to continuity, monotony, or the behaviour in the neighborhood of certain points.

General state of the art. One of the important directions of study over the past decades has been the theory of variational inequalities, whose crucial aspect being the formulation of some conditions of existence for the solutions of such problems. This field began to develop with the work of Fichera [18], who solved a problem formulated by Signorini [30]. The study of the existence and uniqueness of the solution for a problem with variational inequalities has been further developed in various contexts by Stampacchia [31], Browder [9] and Hartman and Stampacchia [19]. Hemivariational inequalities, introduced by Panagiotopoulos [24, 25], are a generalization of these classes of inequalities by replacing the convex functional with a locally Lipschitz functional. For another generalization of these inequalities, set-valued mappings are used in formulating the problems, see, for example, Costea and Lupu [11], Tang *et al.* [34]. An important aspect in proving the theorems regarding the existence of the solution of variational type inequalities is represented by fixed point theorems, see, for example, Costea and Rădulescu [13], Costea and Varga [14], Bai *et al.* [5], Costea and Pitea [12].

The structure of the variational-hemivariational type inequalities combines both convex functionals and locally Lipschitz functions. With regards to the practical applications, they provide the possibility to mathematically model the problems from unilateral contact mechanics.

Another important direction in modern mathematics is represented by the study of best proximity points, that developed as a response for those problems in which

the equation $Tx = x$ has no solutions, where T is a nonself mapping, and x is an element from its domain. Best proximity point theory has its origins in the work of Fan [17] who formulated existence results in the context of a Hausdorff locally convex topological vector space for a continuous mapping. Among the remarkable contributions in this domain we recall the work of: Eldred and Veeramani [16], Suzuki *et al.* [33], Al-Thagafi and Shahzad [1], Basha and Shahzad [8]. Basha [7] introduced the use of the so called proximal contractions for nonempty closed subsets of a complete metric space, in the context of a continuous mapping whose domain and codomain coincide with the two subsets under consideration. The existing studies on best proximity points have considered a wide range of structures, including uniformly convex Banach spaces, reflexive Banach spaces, hyperconvex metric spaces, and various pseudometric spaces; in this regard, we mention the works of Al-Thagafi and Shahzad [1], Aslantas [4], Eldred and Veeramani [16], Kirk *et al.* [22], Petric and Zlatanov [26], Petric [27], Termkaev *et al.* [35], and Zlatanov [36].

Thesis description: structure and content.

This thesis consists of a introductory section, three chapters devoted to presenting the original results, a overview for future directions of study, and a bibliography.

In Chapter 1, **Variational-hemivariational inequality systems**, we investigate a coupled system of variational-hemivariational type inequalities (P) , with variable constraint sets that depend on the solution of the system.

After presenting some preliminary results, mainly concerning set-valued mappings, the problem of study is given as follows:

(P) : Find $u \in K$ and $p \in \Lambda(u)$ such that

$$\begin{cases} \mathcal{C}(v, p) - \mathcal{C}(u, p) + \langle A(u), v - u \rangle + J_1^0(\gamma_1(u), \gamma_2(p); \gamma_1(v - u)) \geq 0, \\ \mathcal{C}(u, q) - \mathcal{C}(u, p) + \langle B(p), q - p \rangle + J_2^0(\gamma_1(u), \gamma_2(p); \gamma_2(q - p)) \geq 0, \end{cases}$$

for all $(v, q) \in K \times \Lambda(u)$, where X, Y, Z_1, Z_2 are real reflexive Banach spaces such that the embeddings $X \hookrightarrow Z_1$ and $Y \hookrightarrow Z_2$ are compact and $K \subseteq X$. Assume that $\Lambda : K \rightrightarrows Y$ is a set-valued mapping with nonempty closed convex values, $\mathcal{C} : X \times Y \rightarrow (-\infty, +\infty]$ is convex and lower semicontinuous with respect to each variable, $J : Z_1 \times Z_2 \rightarrow \mathbb{R}$ is locally Lipschitz with respect to each variable and $A : X \rightarrow X^*, B : Y \rightarrow Y^*$ are prescribed nonlinear operators.

We study the case where the constraint sets K and $\Lambda(u)$, $u \in K$ are bounded.

We impose some adequate hypothesis, as follows:

(H_0) X, Y, Z_1, Z_2 are real reflexive Banach spaces, $0_X \ni K \subseteq X$ is a closed and convex constraint set and the embeddings $X \hookrightarrow Z_1$ and $Y \hookrightarrow Z_2$ are compact;

(H_Λ) $\Lambda : K \rightrightarrows Y$ is set-valued mapping such that:

- (i) Λ has nonempty closed convex values;
- (ii) Λ is lower hemicontinuous from X_w into Y_s ;
- (iii) Λ is upper hemicontinuous from X_w into Y_w ;

(H_C) $\mathcal{C} : X \times Y \rightarrow (-\infty, +\infty]$ is a proper functional such that:

- (i) $K \times \Lambda(K) \subset \text{dom}(\mathcal{C})$;
- (ii) $u \mapsto \mathcal{C}(u, p)$ is convex and lower semicontinuous for all $p \in Y$;
- (iii) $p \mapsto \mathcal{C}(u, p)$ is convex and lower semicontinuous for all $u \in X$;
- (iv) if $\{u_\alpha\}, \{v_\alpha\} \subset K$ and $\{p_\alpha\} \subset \Lambda(K)$ are nets such that $u_\alpha \rightharpoonup u$, $v_\alpha \rightarrow v$ in X and $p_\alpha \rightarrow p$ in Y , then

$$\limsup_{\alpha} [\mathcal{C}(v_\alpha, p_\alpha) - \mathcal{C}(u_\alpha, p_\alpha)] \leq \mathcal{C}(v, p) - \mathcal{C}(u, p);$$

- (v) if $\{u_\alpha\} \subset K$ and $\{p_\alpha\}, \{q_\alpha\} \subset \Lambda(K)$ are nets such that $u_\alpha \rightharpoonup u$ in X , and $p_\alpha \rightarrow p$ and $q_\alpha \rightarrow q$ in Y , then

$$\limsup_{\alpha} [\mathcal{C}(u_\alpha, q_\alpha) - \mathcal{C}(u_\alpha, p_\alpha)] \leq \mathcal{C}(u, q) - \mathcal{C}(u, p).$$

(H_J) $J : Z_1 \times Z_2 \rightarrow \mathbb{R}$ is such that:

- (i) $z_1 \mapsto J(z_1, z_2)$ is locally Lipschitz for all $z_2 \in Z_2$;
- (ii) $z_2 \mapsto J(z_1, z_2)$ is locally Lipschitz for all $z_1 \in Z_1$.

(H_A) $A : X \rightarrow X^*$ is a nonlinear operator such that for any two nets $\{u_\alpha\}, \{w_\alpha\} \subset X$ such that $u_\alpha \rightharpoonup u$ and $w_\alpha \rightharpoonup w$ one has

$$\limsup_{\alpha} \langle A(w_\alpha), v - u_\alpha \rangle \leq \langle A(w), v - u \rangle, \quad \forall v \in X.$$

(H_B) $B : Y \rightarrow Y^*$ is a continuous nonlinear operator, in other words, for any net $\{p_\alpha\} \subset Y$ satisfying $p_\alpha \rightarrow p$ in Y one has $B(p_\alpha) \rightarrow B(p)$ in Y^* .

The study is made using an auxiliary problem denoted by $\mathcal{P}_{aux}(\omega, r, \xi, \eta)$, which is formulated using adjoint operators:

$\mathcal{P}_{aux}(w, r, \xi, \eta)$: Find $(u, p) \in K \times \Lambda(w)$ such that

$$\begin{aligned} \mathcal{C}(v, r) - \mathcal{C}(u, r) + \mathcal{C}(w, q) - \mathcal{C}(w, p) + \langle A(w) + \gamma_1^*(\xi), v - u \rangle \\ + \langle B(r) + \gamma_2^*(\eta), q - p \rangle \geq 0, \end{aligned}$$

for all $(v, q) \in K \times \Lambda(w)$.

In Lemma 1, we show that the solution set of $\mathcal{P}_{aux}(\omega, r, \xi, \eta)$ is nonempty, weakly compact and convex. Based on this result, we enunciate Theorem 1, in which we study the weak compactness of the solution set of problem (P), also using the Bohnenblust-Karlin fixed point theorem.

Lemma 1. Assume (H_0) , (H_C) , (H_Λ) , (H_A) , (H_B) hold. In addition, suppose $K \subset X$ is bounded and Λ has bounded values. Then the solution mapping $\mathcal{S} : K \times Y \times Z_1^* \times Z_2^* \rightharpoonup K \times Y$ defined by

$$\mathcal{S}(w, r, \xi, \eta) := \{(u, p) \in K \times \Lambda(w) : (u, p) \text{ solves } \mathcal{P}_{aux}(w, r, \xi, \eta)\}$$

has nonempty, weakly compact and convex values.

Theorem 1. Assume (H_0) , (H_Λ) , (H_C) , (H_J) , (H_A) and (H_B) hold. If $K \subset X$ bounded and Λ has bounded values, then the solution set of problem (P) is nonempty and weakly compact.

In the next part of this chapter, we study the case in which at least one of the constraint sets is unbounded, as follows:

- If K is unbounded and Λ has bounded values, then we assume (C_c^1) :

$$(i) \quad \sup_{(u,p) \in K \times \Lambda(K)} L_1(u, p) < \infty, \text{ where } L_1(u, p) \text{ is the Lipschitz constant near the point } (\gamma_1(u), \gamma_2(p)) \in Z_1 \times Z_2 \text{ of the mapping } z_1 \mapsto J(z_1, z_2);$$

$$(ii) \quad \lim_{\substack{u \in K \\ \|u\|_X \rightarrow \infty}} \frac{\langle A(u), u \rangle + \mathcal{C}(u, p) - \mathcal{C}(0_X, p)}{\|u\|_X} = \infty, \forall p \in \Lambda(u).$$

- If K is bounded and Λ has unbounded values, then we assume (C_c^2) :

$$(i) \quad \sup_{(u,p) \in K \times \Lambda(K)} L_2(u, p) < \infty, \text{ where } L_2(u, p) \text{ is the Lipschitz constant near the point } (\gamma_1(u), \gamma_2(p)) \in Z_1 \times Z_2 \text{ of the mapping } z_2 \mapsto J(z_1, z_2);$$

(ii) For all $u \in K$ one has

$$\lim_{\substack{p \in \Lambda(u) \\ \|p\|_Y \rightarrow \infty}} \frac{\langle B(p), p \rangle + \mathcal{C}(u, p) - \mathcal{C}(u, 0_Y)}{\|p\|_Y} = \infty.$$

• If K is unbounded and Λ has unbounded values, then we assume (C_c^3) :

$$(i) \max \left\{ \sup_{(u,p) \in K \times \Lambda(K)} L_1(u, p), \sup_{(u,p) \in K \times \Lambda(K)} L_2(u, p) \right\} < \infty;$$

$$(ii) \lim_{\|(u,p)\| \rightarrow \infty} \frac{\langle A(u), u \rangle + \langle B(p), p \rangle + 2\mathcal{C}(u, p) - \mathcal{C}(0_X, p) - \mathcal{C}(u, 0_Y)}{\|(u, p)\|} = \infty.$$

In Theorems 2, 3 and 4 we give sufficient conditions for the existence of the solution of problem (P) , for the cases mentioned above.

Theorem 2. Assume K is unbounded, Λ has bounded values and (H_0) , (H_Λ) , (H_C) , (H_J) , (H_A) , (H_B) and (C_c^1) hold. Then the solution of problem (P) is nonempty.

Theorem 3. Assume K is bounded, Λ has unbounded values and $0_Y \in \Lambda(u)$, for all $u \in K$. If (H_0) , (H_Λ) , (H_C) , (H_J) , (H_A) , (H_B) and (C_c^2) hold, then the solution of problem (P) is nonempty.

Theorem 4. Assume K is unbounded, Λ assumes unbounded values and $0_Y \in \Lambda(u)$, for all $u \in K$. If (H_0) , (H_Λ) , (H_C) , (H_J) , (H_A) , (H_B) and (C_c^3) hold, then the solution of problem (P) is nonempty.

In the last part of this chapter, we study an application from nonsmooth mechanics, more precisely, we consider a mathematical model $(\mathcal{P}_M)$, which describes the contact between a deformable elastic body and a Winkler type foundation, under the assumption that the process is static and the behaviour of the material is described by a possibly multivalued constitutive law written as a subdifferential inclusion.

Find a displacement field $\vec{u} : \bar{\Omega} \rightarrow \mathbb{R}^3$ and stress tensor $\sigma : \bar{\Omega} \rightarrow \mathbb{S}_3$ such that

$$(\mathcal{P}_M) : \left\{ \begin{array}{ll} -\text{Div}(\sigma) = \vec{f}_0, & \text{in } \Omega, \\ \sigma \in \mathcal{E}\varepsilon(\vec{u}) + \partial\psi(\varepsilon(\vec{u})), & \text{in } \Omega, \\ \vec{u} = \vec{0}, & \text{on } \Gamma_1, \\ \sigma \vec{\nu} = \vec{f}_2, & \text{on } \Gamma_2, \\ \left. \begin{array}{l} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu \in [0, a(x)) \Rightarrow -\sigma_\nu = \phi(x, u_\nu), \\ u_\nu = a(x) \Rightarrow -\sigma_\nu \in [0, \phi(x, a(x))], \\ u_\nu > a(x) \Rightarrow \sigma_\nu = 0, \\ \vec{\sigma}_\tau = \vec{S}_\tau, \end{array} \right\} & \text{on } \Gamma_3. \end{array} \right.$$

Suppose that the following conditions for problem $(\mathcal{P}_M)$ are fulfilled:

$$(\mathcal{H}_f) \quad \vec{f}_0 \in L^2(\Omega; \mathbb{R}^3), \vec{f}_2 \in L^2(\Gamma_2; \mathbb{R}^3), \vec{S}_\tau \in L^2(\Gamma_3; \mathbb{R}^3).$$

$(\mathcal{H}_\mathcal{E})$ $\mathcal{E} : \Omega \times \mathbb{S}_3 \rightarrow \mathbb{S}_3$ is fourth-order tensor such that

$$(i) \quad \mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq 3;$$

$$(ii) \quad \mathcal{E}\sigma : \mu = \sigma : \mathcal{E}\mu, \text{ for all } \sigma, \mu \in \mathbb{S}_3;$$

$$(iii) \quad \text{there exists } m_\mathcal{E} > 0 \text{ such that } \mathcal{E}\mu : \mu \geq m_\mathcal{E}|\mu|^2, \text{ for all } \mu \in \mathbb{S}_3.$$

$(\mathcal{H}_\psi)$ $\psi : \mathbb{S}_3 \rightarrow (-\infty, +\infty]$ is a proper convex lower semicontinuous functional such that

$$(i) \quad \psi(\mathbf{0}) \in L^1(\Omega);$$

$$(ii) \quad \text{dom}(\psi) \cap \mathcal{H} \neq \emptyset.$$

$$(\mathcal{H}_a) \quad 0 < a(\cdot) \in L^\infty(\Gamma_3);$$

$(\mathcal{H}_\phi)$ $\phi : \Gamma_3 \times [0, \infty) \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$(i) \quad \phi(\cdot, t) \text{ is measurable for all } t \in \mathbb{R};$$

$$(ii) \quad \phi(x, 0) = 0 \text{ for almost all } x \in \Gamma_3;$$

$$(iii) \quad t \mapsto \phi(x, \cdot) \text{ is strictly increasing on } \mathbb{R} \text{ for almost all } x \in \Gamma_3.$$

In this chapter, we also enunciate the primal-dual variational formulation $(\mathcal{P}_{p-d})$, together with the primal variational formulation $(\mathcal{P}_p)$ and the dual variational formulation $(\mathcal{P}_d(\vec{u}))$ of problem $(\mathcal{P}_M)$, as follows:

$(\mathcal{P}_{p-d})$: Find a displacement field $\vec{u} \in K$ and a stress tensor $\sigma \in \Lambda(\vec{u})$ such that

$$\begin{cases} G(\varepsilon(\vec{v})) - G(\varepsilon(\vec{u})) + \int_{\Gamma_3} j_{,2}^0(x, u_\nu; v_\nu - u_\nu) da \geq \left(\vec{f}, \vec{v} - \vec{u} \right)_X, \quad \forall \vec{v} \in K, \\ G^*(\mu) - G^*(\sigma) - (\varepsilon(\vec{u}), \mu - \sigma)_\mathcal{H} \geq 0, \quad \forall \mu \in \Lambda(\vec{u}). \end{cases}$$

$(\mathcal{P}_p)$: Find $\vec{u} \in K$ such that

$$(\mathcal{F}(\vec{u}), \vec{v} - \vec{u})_X + \Psi(\vec{v}) - \Psi(\vec{u}) + \int_{\Gamma_3} j_{,2}^0(x, u_\nu; v_\nu - u_\nu) da \geq \left(\vec{f}, \vec{v} - \vec{u} \right)_X,$$

for all $\vec{v} \in K$.

$(\mathcal{P}_d(\vec{u}))$: Find a stress tensor $\sigma \in \mathcal{H}$ such that

$$\mathbf{0} \in \partial \left(G^* + I_{\Lambda(\vec{u})} \right) (\sigma).$$

In Theorem 5, it can be observed the connection between the variational formulation types.

Theorem 5. Assume $(\mathcal{H}_f)$, $(\mathcal{H}_\varepsilon)$, $(\mathcal{H}_\psi)$, $(\mathcal{H}_a)$ and $(\mathcal{H}_\phi)$ hold. If a pair $(\vec{u}, \sigma)$ is a solution of $(\mathcal{P}_{p-d})$, then $\vec{u}$ solves $(\mathcal{P}_p)$ and σ solves $(\mathcal{P}_d(\vec{u}))$.

In Theorem 6, we study the weak solvability of problem $(\mathcal{P}_{p-d})$.

Theorem 6. Assume $(\mathcal{H}_f)$, $(\mathcal{H}_\varepsilon)$, $(\mathcal{H}_\psi)$, $(\mathcal{H}_a)$ and $(\mathcal{H}_\phi)$ hold. Then problem $(\mathcal{P}_M)$ has at least one weak solution, that is, the solution set of problem $(\mathcal{P}_{p-d})$ is nonempty.

The chapter is based on article [10].

In chapter 2, **Proximal type mappings**, we study some conditions in which the distance from a point x to its image through $\mathcal{T}: A \rightarrow B$ it reaches its minimum possible value, that is, the distance between the sets A and B , where A, B are nonempty subsets of a set X , endowed with a metric type structure, while also being fulfilled additional restrictions. In this chapter, we further generalize the study made in Ali *et al.* [2] and Ali *et al.* [3], respectively.

In the first part of the chapter we define some preliminary concepts and properties in the context of metric spaces, which are crucial for the development of the main results of this study. In Definition 2 are given proximal contractive conditions of type $\mathcal{L}$ by using the class of functions from Definition 1.

Definition 1. Denote by $\mathfrak{M}$ the class of functions $L: [0, \infty)^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{R}$ which fulfill the following hypotheses:

(M_1) for each $\{x_n | x_n \geq 0\}$, $\{y_n | y_n \geq 0\}$, $\{z_n | z_n \geq 0\}$, we have

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$$

if and only if

$$\lim_{n \rightarrow \infty} L(x_n, y_n, z_n) = -\infty;$$

(M_2) for each $\{x_n | x_n \geq 0\}$, $\{y_n | y_n \geq 0\}$, $\{z_n | z_n \geq 0\}$, with $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$, there exists $h \in (0, 1)$ so that $\lim_{n \rightarrow \infty} x_n^h L(x_n, y_n, z_n) = 0$.

Denote by $\mathfrak{M}_b$ the class of functions that respects the hypothesis (M_1) .

Definition 2. Let (X, d) be a metric space and A, B be two nonempty subsets of X . A mapping $\mathcal{T}: A \rightarrow B$ is called $\mathcal{L}$ proximal contraction if there exist a function $L \in \mathfrak{M}$ and a constant $k > 0$ such that for all $\xi_1, \xi_2, \zeta_1, \zeta_2 \in A$, with $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, we have

$$k + L(d(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2)) \leq L(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2)), \quad (1)$$

whenever

$$\min\{d(\zeta_1, \zeta_2) + \phi(\zeta_1) + \varphi(\zeta_2), d(\xi_1, \xi_2) + \phi(\xi_1) + \varphi(\xi_2)\} > 0.$$

Theorem 7 enunciates the first result for the existence of (ϕ, φ) -best proximity points, by using the proximal contractions of type $\mathcal{L}$.

Theorem 7. *Let A and B be nonempty subsets of X , and (X, d) be a complete metric space. Consider that A_0 is closed with respect to the metric d and a $\mathcal{L}$ proximal contraction mapping $\mathcal{T}: A \rightarrow B$, which fulfills the following conditions:*

- (i) $\mathcal{T}A_0 \subseteq B_0$;
- (ii) ϕ and φ are lower semicontinuous mappings.

Then $\mathcal{T}$ has a (ϕ, φ) -best proximity point.

Another family of weak proximal contractions is introduced in Definition 5, formulated by using the classes of functions $\mathfrak{L}_w$ and $\mathfrak{W}$, respectively.

Definition 3. Denote by $\mathfrak{L}_w$ the class of functions $L: (0, \infty) \rightarrow \mathbb{R}$ which satisfies the next axioms:

- (L_1) L is increasing, that is, if $a_1 \leq a_2$ then $L(a_1) \leq L(a_2)$;
- (L_2) for each sequence $\{c_n \mid c_n > 0\}$ with $\lim_{n \rightarrow \infty} c_n = 0$, there exists $h \in (0, 1)$ such that $\lim_{n \rightarrow \infty} c_n^h L(c_n) = 0$.

We use the set $\mathcal{C} = \{(0, x, y) \mid x, y \geq 0\}$ to introduce the class of functions $\mathfrak{W}$.

Definition 4. Denote by $\mathfrak{W}$ the class of functions $W: [0, \infty)^3 \rightarrow [0, \infty)$ which satisfy the next hypotheses:

- (W_1) W is continuous on $\mathcal{C}$ and $W(x, y, z) = 0$ if and only if $x = y = z = 0$;
- (W_2) there is $\beta > 0$ with $\beta a \leq W(a, b, c)$, for any $a, b, c \geq 0$.

Definition 5. A mapping $\mathcal{T}: A \rightarrow B$ is called LW proximal contraction if there exists the functions $\alpha: A \times A \rightarrow [0, \infty)$, $L \in \mathfrak{L}_w$, $W \in \mathfrak{W}$, and a constant $k > 0$ so that for all $\xi_1, \xi_2, \zeta_1, \zeta_2 \in A$, with $\alpha(\xi_1, \xi_2) \geq 1$ and $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, we get $\alpha(\zeta_1, \zeta_2) \geq 1$ and

$$k + L(W(d(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2))) \leq L(W(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2))), \quad (2)$$

whenever

$$\min\{W(d(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2)), W(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2))\} > 0.$$

The second theorem for the existence of best proximity points, Theorem 8, is formulated using the proximal contractions of type LW .

Theorem 8. *Let A and B be nonempty subsets of X , and (X, d) a complete metric space. Consider that A_0 is closed with respect to d , and that $\mathcal{T}: A \rightarrow B$ is a LW proximal contraction mapping. Suppose that the following conditions hold:*

- (i) $\mathcal{T}A_0 \subseteq B_0$;
- (ii) there are $\xi_0, \xi_1 \in A_0$, so that $\alpha(\xi_0, \xi_1) \geq 1$ and $d(\xi_1, \mathcal{T}\xi_0) = d(A, B)$;
- (iii) every sequence $\{\xi_n\}$ from A_0 with $\xi_n \rightarrow \xi$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$, $n \in \mathbb{N}$, satisfies the inequality $\alpha(\xi_n, \xi) \geq 1$, $n \in \mathbb{N}$;
- (iv) the functions $\phi, \varphi: A \rightarrow [0, \infty)$ are bounded and lower semi continuous.

Then $\mathcal{T}$ has a (ϕ, φ) -best proximity point.

The second part of this chapter is developed in the context of b -metric spaces whose coefficient is strictly greater than one.

The notion of b -metric is attributed in literature to Bakhtin [6] and Czerwik [15].

Definition 6. Let X be a nonempty set and $s \geq 1$ a real number. A function $d: X \times X \rightarrow [0, \infty)$ is called a b -metric if the following conditions are satisfied, for every $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s [d(x, z) + d(z, y)]$.

In this case, (X, d) is called a b -metric space with a constant $s \geq 1$.

It is mentioned that the class of b -metrics properly contains that of classic metric spaces, with the constant $s = 1$. Furthermore, continuity is not, in general, a feature of b -metrics.

To prove the next theorem of this chapter, it is necessary to introduce an additional condition, stated in Definition 7, inspired by the work of Lucács și Kajántó [23].

Definition 7. Let (X, d) be a b -metric space with the constant s and $A \subseteq X$ be a nonempty set. We say that $L: [0, \infty)^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{R}$ satisfies the property (S_L) , with respect to $\phi, \varphi: A \rightarrow [0, \infty)$, if the next implication holds

(S_L) If there is $k > 0$ and $\{\xi_n\}$ a sequence from A such that $d(\xi_n, \xi_{n+1}) > 0$, $n \in \mathbb{N}$, and

$$k + L(sd(\xi_n, \xi_{n+1}), \phi(\xi_n), \varphi(\xi_{n+1})) \leq L(d(\xi_{n-1}, \xi_n), \phi(\xi_{n-1}), \varphi(\xi_n)),$$

for all $n \geq 1$, then

$$k + L(s^n d(\xi_n, \xi_{n+1}), \phi(\xi_n), \varphi(\xi_{n+1})) \leq L(s^{n-1} d(\xi_{n-1}, \xi_n), \phi(\xi_{n-1}), \varphi(\xi_n)),$$

for all $n \geq 1$.

The class of proximal contractive operators of type L_b , which can be found in Definition 8, is enunciated using the class of functions $\mathfrak{M}_b$.

Definition 8. Let (X, d) be a b -metric space and A and B be two nonempty subsets of X . A mapping $\mathcal{T}: A \rightarrow B$ is called L_b proximal contraction if there exists the function $L \in \mathfrak{M}_b$ and a constant $k > 0$ such that for all $\xi_1, \xi_2, \zeta_1, \zeta_2 \in A$, with $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, the following inequality holds

$$k + L(sd(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2)) \leq L(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2)), \quad (3)$$

whenever

$$\min\{sd(\zeta_1, \zeta_2) + \phi(\zeta_1) + \varphi(\zeta_2), d(\xi_1, \xi_2) + \phi(\xi_1) + \varphi(\xi_2)\} > 0.$$

Using the L_b proximal contraction and the property (S_L) , we enunciate, in Theorem 9, some conditions for the existence of best proximity points.

Theorem 9. Let A and B be non-void subsets of X , and (X, d) be a complete b -metric space. Consider that A_0 is closed with respect to d , with $s > 1$, and that $\mathcal{T}: A \rightarrow B$ is a L_b proximal contraction mapping, which fulfills the next conditions:

- (i) $\mathcal{T}A_0 \subseteq B_0$;
- (ii) the property (S_L) is satisfied with respect to ϕ, φ ;
- (iii) the functions ϕ și φ are lower semi continuous.

Then $\mathcal{T}$ has a (ϕ, φ) -best proximity point.

Definition 9 presents the property (S_{LW}) , which plays a crucial role in the next best proximity point theorem.

Definition 9. Let (X, d) be a b -metric space with the constant s and $A \subseteq X$ be a nonempty set. We say that $L: (0, \infty) \rightarrow \mathbb{R}$ and $W: [0, \infty)^3 \rightarrow [0, \infty)$ satisfies the property (S_{LW}) , with respect to $\phi, \varphi: A \rightarrow [0, \infty)$, if the next implication holds

(S_{LW}) If there is $k > 0$ and $\{\xi_n\} \subset A$ a sequence such that

$$k + L(W(sd(\xi_n, \xi_{n+1}), \phi(\xi_n), \varphi(\xi_{n+1}))) \leq L(W(d(\xi_{n-1}, \xi_n), \phi(\xi_{n-1}), \varphi(\xi_n))),$$

for all $n \geq 1$, then

$$k + L(W(s^n d(\xi_n, \xi_{n+1}), \phi(\xi_n), \varphi(\xi_n))) \leq L(W(s^{n-1} d(\xi_{n-1}, \xi_n), \phi(\xi_{n-1}), \varphi(\xi_n))),$$

for all $n \geq 1$.

Using the class of functions $\mathfrak{W}$ in Definition 10, it is introduced a new type of proximal contraction mappings.

Definition 10. Let (X, d) be a b -metric space. Consider A and B two nonempty subsets of X , $L: (0, \infty) \rightarrow \mathbb{R}$ an increasing function and $W \in \mathfrak{W}$. A mapping $\mathcal{T}: A \rightarrow B$ is called LW_b proximal contraction if there is a constant $k > 0$ so that for any $\xi_1, \xi_2, \zeta_1, \zeta_2 \in A$, with $\alpha(\xi_1, \xi_2) \geq 1$ and $d(\zeta_1, \mathcal{T}\xi_1) = d(A, B) = d(\zeta_2, \mathcal{T}\xi_2)$, we get $\alpha(\zeta_1, \zeta_2) \geq 1$ and

$$k + L(W(sd(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2))) \leq L(W(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2))), \quad (4)$$

whenever

$$\min\{W(sd(\zeta_1, \zeta_2), \phi(\zeta_1), \varphi(\zeta_2)), W(d(\xi_1, \xi_2), \phi(\xi_1), \varphi(\xi_2))\} > 0.$$

Theorem 10 provides conditions for the existence of best proximity points using Definition 9 and Definition 10.

Theorem 10. Let A and B be nonempty subsets of X , and (X, d) be a complete b -metric space, with $s > 1$. Consider that A_0 is nonempty and closed with respect to d , and that $\mathcal{T}: A \rightarrow B$ is a LW_b proximal contraction mapping, which fulfills the following conditions:

- (i) $\mathcal{T}A_0 \subseteq B_0$;
- (ii) there are $\xi_0, \xi_1 \in A_0$, so that $\alpha(\xi_0, \xi_1) \geq 1$ and $d(\xi_1, \mathcal{T}\xi_0) = d(A, B)$;
- (iii) every sequence $\{\xi_n\}$ from A_0 with $\xi_n \rightarrow \xi$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, necessarily satisfies the inequality $\alpha(\xi_n, \xi) \geq 1$, for all $n \in \mathbb{N}$;
- (iv) the property (S_{LW}) is satisfied;
- (v) the functions $\phi, \varphi: A \rightarrow [0, \infty)$ are bounded lower semi continuous.

Then $\mathcal{T}$ has a (ϕ, φ) -best proximity point.

The chapter concludes with the case when $\mathcal{T}: X \rightarrow X$ is a self mapping, which provides fixed point results.

Therem 11 and Theorem 12, respectively, enunciate fixed point consequences in the context of classic metric spaces.

Theorem 11. *Let (X, d) be a complete metric space. Consider that $\mathcal{T}: X \rightarrow X$ is a mapping which fulfills the following conditions:*

(i) *there is $L \in \mathfrak{M}$ and $k > 0$ so that*

$$k + L(d(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta)) \leq L(d(\xi, \zeta), \phi(\xi), \varphi(\zeta)),$$

whenever $\min\{d(\mathcal{T}\xi, \mathcal{T}\zeta) + \phi(\mathcal{T}\xi) + \varphi(\mathcal{T}\zeta), d(\xi, \zeta) + \phi(\xi) + \varphi(\zeta)\} > 0$, when $\xi, \zeta \in X$;

(ii) *ϕ and φ are lower semi continuous functions.*

Then $\mathcal{T}$ has a (ϕ, φ) -fixed point.

Theorem 12. *Let (X, d) be a complete metric space. Consider that $\mathcal{T}: X \rightarrow X$ is a mapping which fulfills the following conditions:*

(i) *there exists $L \in \mathfrak{L}_w$, $W \in \mathfrak{W}$ and a constant $k > 0$ so that*

$$k + L(W(d(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta))) \leq L(W(d(\xi, \zeta), \phi(\xi), \varphi(\zeta))),$$

whenever $\min\{W(d(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta)), W(d(\xi, \zeta), \phi(\xi), \varphi(\zeta))\} > 0$, when $\xi, \zeta \in X$;

(ii) *ϕ and φ are bounded lower semi continuous functions.*

Then $\mathcal{T}$ has a (ϕ, φ) -fixed point.

Regarding the case of b -metric spaces, Theorem 13 and Theorem 14, respectively, enunciate consequences of the previous best proximity point results.

Theorem 13. *Let (X, d) be a complete b -metric space, with $s > 1$. Consider that $\mathcal{T}: X \rightarrow X$ is a self mapping which satisfies the following conditions:*

(i) *there is $L \in \mathfrak{M}_b$ and $k > 0$ so that*

$$k + L(sd(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta)) \leq L(sd(\xi, \zeta), \phi(\xi), \varphi(\zeta)),$$

whenever $\min\{sd(\mathcal{T}\xi, \mathcal{T}\zeta) + \phi(\mathcal{T}\xi) + \varphi(\mathcal{T}\zeta), sd(\xi, \zeta) + \phi(\xi) + \varphi(\zeta)\} > 0$, $\xi, \zeta \in X$.

(ii) the property (S_L) is satisfied with respect to ϕ, φ ;

(iii) ϕ and φ are lower semi continuous functions.

Then $\mathcal{T}$ has a (ϕ, φ) -fixed point.

Theorem 14. Let (X, d) be a complete b -metric space, with $s > 1$. Consider that $\mathcal{T}: X \rightarrow X$ is a self mapping which satisfies the following conditions:

(i) there exists $L: (0, \infty) \rightarrow \mathbb{W}$ a increasing function, $W \in \mathfrak{R}$ and a constant $k > 0$ so that

$$k + L(W(sd(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta))) \leq L(W(d(\xi, \zeta), \phi(\xi), \varphi(\zeta))),$$

whenever $\min\{W(sd(\mathcal{T}\xi, \mathcal{T}\zeta), \phi(\mathcal{T}\xi), \varphi(\mathcal{T}\zeta)), W(d(\xi, \zeta), \phi(\xi), \varphi(\zeta))\} > 0$, $\xi, \zeta \in X$;

(ii) the property (S_{LW}) is satisfied with respect to ϕ, φ ;

(iii) ϕ and φ are bounded lower semi continuous functions.

Then $\mathcal{T}$ has a (ϕ, φ) -fixed point.

The chapter is based on article [28].

In the chapter **Best proximity points on generalized metric spaces**, we extend the results introduced by Istrăţescu [20], both from the point of view of the object of study, by considering best proximity points, and also by working on a generalized metric space, inspired by the work of Jleli and Samet [21], which can be found in Definition 11.

Definition 11 ([21]). Let X be a nonempty set and let $d: X \times X \rightarrow [0, \infty)$ be a function which satisfies the following conditions:

d_1) if $d(x, y) = 0$, then $x = y$, when $x, y \in X$;

d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

d_3) there exists $C > 0$ such that for every $x, y \in X$, and any sequence $\{x_n\}$ with the property that $d(x_n, x) \rightarrow 0$, it follows that

$$d(x, y) \leq C \limsup_{n \rightarrow \infty} d(x_n, y).$$

In this chapter it is used a generalized metric space (X, d) , where function d satisfies the conditions from Definition 11, and also $d(x, x) = 0$, for any $x \in X$.

It is pointed out that a number of classes of metric spaces can be seen as particular cases of this one, from which we mention classic metric spaces, b -metric spaces, or modular spaces with the Fatou property.

In the context of best proximity points, the following set

$$\mathcal{O}_T(x) = \{\{x_n\} \subseteq A \mid x_0 = x, d(x_{n+1}, Tx_n) = d(A, B)\},$$

is called the orbit of x under T , when (X, d) is a generalized metric space, A and B are nonempty subsets of X and $T: A \rightarrow B$.

The point $x \in X$ is called a periodic best proximity point for T if there exists $\{x_n\} \in \mathcal{O}_T(x)$ and $m \geq 1$ so that $d(x, Tx_m) = d(A, B)$. The mapping T satisfies condition P_b if the set of periodic best proximity points coincides with the set of best proximity points for T .

In this chapter, it is used a mapping T that satisfies property P_b (see, for example, Sahin *et al.* [29]).

In Theorem 15, a boundedness condition is imposed on certain sequences defined using pairs of proximal subsets, as follows.

Let $\{x_n\}$ be a sequence from X , where (X, d) is a generalized metric space. Consider the sets

$$\delta(d, \{x_n\}) = \sup\{d(x_m, x_p) \mid m, p \in \mathbb{N}\},$$

and

$$\delta_{n_0}(d, \{x_n\}) = \sup\{d(x_m, x_p) \mid m, p \in \mathbb{N}, m, p \geq n_0\},$$

where $n_0 \in \mathbb{N}$.

By the property of symmetry, it follows that

$$\delta_{n_0}(d, \{x_n\}) = \sup\{d(x_m, x_p) \mid m, p \in \mathbb{N}, p \geq m \geq n_0\}.$$

It is concluded that

$$\delta_p(d, \{x_n\}) \leq \delta_m(d, \{x_n\}) \leq \delta(d, \{x_n\}), p \geq m.$$

Theorem 15. *Let (X, d) be a complete generalized metric space, and $T: A \rightarrow B$ be a nonself mapping with $TA_0 \subseteq B_0$.*

Suppose the following conditions are fulfilled:

i) there are $a_1, a_2, \dots, a_{p-1}, b_1, \dots, b_{p-2} \in [0, 1)$, $0 < a_1 + a_2 + \dots + a_{p-1} + b_1 + \dots + b_{p-2} < 1$, $p \geq 3$, so that for any distinct z_i , $i \in \mathbb{N}$, and for any $y_i \in A$ with $d(y_i, Tz_i) = d(A, B)$, we have

$$\begin{aligned} d(y_{p-1}, y_p) &\leq a_1 d(z_1, z_2) + a_2 d(z_2, z_3) + \dots + a_{p-1} d(z_{p-1}, z_p) \\ &\quad + b_1 d(y_1, y_2) + \dots + b_{p-2} d(y_{p-2}, y_{p-1}); \end{aligned} \tag{5}$$

- ii) A_0 is closed with respect to d ;
- iii) for any sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A, B)$, $n \in \mathbb{N}$, we have $\delta(d, \{x_n\}) < \infty$.

Then T has a unique best proximity point.

Corollary 1 and Corollary 2, respectively, enunciates interesting consequences from literature for Theorem 15.

Corollary 1. Let (X, d) be a complete generalized metric space, and $T: A \rightarrow B$ be a nonself mapping with $TA_0 \subseteq B_0$.

Suppose the following conditions are fulfilled:

- i) there are $a, b \in [0, 1)$, $0 < a + b < 1$, so that for any distinct $x_i \in A$, and for any $y_i \in A$, with $d(y_i, Tx_i) = d(A, B)$, $i \in \mathbb{N}$, we have

$$d(y_2, y_3) \leq ad(x_1, x_2) + bd(x_2, x_3). \quad (6)$$

- ii) A_0 is closed with respect to d ;
- iii) for any sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A, B)$, $n \in \mathbb{N}$, we have $\delta(d, \{x_n\}) < \infty$.

Then T has a unique best proximity point.

Corollary 2. Let (X, d) be a complete generalized metric space, and $T: A \rightarrow B$ be a nonself mapping with $TA_0 \subseteq B_0$.

Suppose the following conditions are fulfilled:

- i) there are $a, b \in [0, 1)$, $0 < a + b < 1$, so that for any distinct $x_i \in A$, and for any $y_i \in A$, with $d(y_i, Tx_i) = d(A, B)$, $i \in \mathbb{N}$, we have

$$d(y_2, y_3) \leq ad(x_1, x_2) + bd(y_1, y_2); \quad (7)$$

- ii) A_0 is closed with respect to d ;
- iii) for any sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A, B)$, $n \in \mathbb{N}$, we have $\delta(d, \{x_n\}) < \infty$.

Then T has a unique best proximity point.

Unlike the results from Theorem 15, Theorem 16 does not require any boundedness conditions, however, the constant of the generalized metric space must take values from a certain set.

Theorem 16. Let (X, d) be a generalized metric space with the constant C such that $Ca_p < 1$, and $T: A \rightarrow B$ be a nonself mapping so that $TA_0 \subseteq B_0$.

Suppose the following conditions are fulfilled:

i) there are $a_1, a_2, \dots, a_p \in [0, 1)$, $0 < a_1 + a_2 + \dots + a_p < 1$, $p \geq 2$, so that for any distinct $z_i \in A$, $i = \overline{1, p}$ and for any $y_i \in A$, with $d(y_i, Tz_i) = d(A, B)$, $i \in \mathbb{N}$, we have

$$d(y_{p-1}, y_p) \leq a_1 d(z_1, y_1) + a_2 d(z_2, y_2) + \dots + a_p d(z_p, y_p), \quad (8)$$

ii) A_0 is closed with respect to d .

Then T has a unique best proximity point.

Based on Theorem 16, results already existing in literature are obtained, as presented in Corollary 3.

Corollary 3. Let (X, d) be a generalized metric space with the constant $0 < C < 1$ and $T: X \rightarrow X$ be a self mapping.

Suppose there are $a, b \in [0, 1)$, $0 < a + b < 1$, so that for any x, y from X , the following relation holds

$$d(Tx, Ty) \leq ad(y, Ty) + bd(x, Tx). \quad (9)$$

Then T has a unique fixed point.

The chapter is based on article [32].

References

1. Al-Thagafi MA, Shahzad N, Convergence and existence results for best proximity points, *Nonlinear Anal.*, 70, 2009, 3665-3671
2. Ali MU, Ionescu C, **Stanciu M**, On (ϕ, φ) -best proximity points for proximal type contraction mappings, *U. Politeh. Buch. Sci. Bull. Ser. A*, 83, 2021, 15-26
3. Ali MU, Pitea A, **Stanciu M**, On the existence of (η, ζ) -fixed points and (η, ζ) -best proximity points for Wardowski type mappings, *J. Nonlinear Convex Anal.*, 23, 2022, 713-725
4. Aslantas M, A new contribution to best proximity point theory on quasi metric spaces and an application to nonlinear integral equations, *Optimization*, 72, 2023, 2851-2864
5. Bai YR, Migórski S, Nguyen VT, Peng JW, Existence of solution to a new class of coupled variational-hemivariational inequalities, *J. Nonlinear Var. Anal.*, 6, 2022, 499-516
6. Bakhtin IA, The contraction mapping principle in almost metric spaces, *Funct. Anal.*, 30, 1989, 26-37
7. Basha S, Best proximity point theorems, *J. Approx. Theory*, 163, 2011, 1772-1781
8. Basha S, Shahzad N, Best proximity point theorems for generalized proximal contractions, *Fixed Point Theory Appl.*, 2012, art. No. 2012:42
9. Browder F, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.*, 71, 1965, 780-785
10. Cen J, Costea N, Dumitrescu D, **Stanciu M**, A set-valued approach for quasi-variational-hemivariational inequality systems, *J. Fixed Point Theory Appl.*, 27, 2025, 64

11. Costea N, Lupu C, On a class of variational-hemivariational inequalities involving set-valued mappings, *Adv. Pure Appl. Math.*, 1, 2010, 233-246
12. Costea N, Pitea A, Existence results for mixed hemivariational-like inequalities involving set-valued maps, *Optimization*, 70, 2021, 269-305
13. Costea N, Rădulescu V, Inequality problems of quasi-hemivariational type involving set-valued operators and a nonlinear term, *J. Glob. Optim.*, 52, 2012, 743-756
14. Costea N, Varga C, Systems of nonlinear hemivariational inequalities and applications, *Topol. Methods Nonlinear Anal.*, 41, 2013, 39-65
15. Czerwik S, Contraction mappings in b -metric spaces, *Acta Math. Inform. Univ. Ostrav.*, 1, 1993, 5-11
16. Eldred A, Veeramani P, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, 323, 2006, 1001-1006
17. Fan K, Extensions of two fixed point theorems of F.E. Browder, *Math. Z.*, 112, 1969, 234-240
18. Fichera G, Problemi elettrostatici con vincoli unilaterali: il problema de Signorini con ambigue condizioni al contorno, *Mem. Acad. Naz. Lincei*, 7, 1964, 91-140
19. Hartman GJ, Stampacchia G, On some nonlinear elliptic differential equations, *Acta Math.*, 112, 1966,
20. Istrăţescu VI, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters I, *Ann. Mat. Pur. Appl.*, 130, 1982, 89-104
21. Jleli M, Samet B, A generalized metric space and related fixed point theorems, *J. Fixed Point Theory Appl.*, 61, 2015
22. Kirk WA, Reich S, Veeramani P, Proximinal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.*, 24, 2003, 851-862
23. Lucács A, Kajántó S, Fixed point theorems for various types of F -contractions in complete b -metric space, *Fixed Point Theory*, 19, 2018, 321-334
24. Panagiotopoulos PD, Nonconvex energy functions, hemivariational inequalities and substationarity principles, *Acta Mech.*, 42, 1983, 160-183

25. Panagiotopoulos PD, Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions, Birkhäuser, 1985
26. Petric M, Zlatanov B, Best proximity points for p -cyclic summing iterated contractions, Filomat, 32, 2018, 3275-3287
27. Petric M, Best proximity point theorems for weak cyclic Kannan contractions, Filomat, 25, 2011, 145-154
28. Pitea A, **Stanciu M**, Best proximity for proximal operators on b -metric spaces, Turkish J. Math., 47, 2023, 1873-1886
29. Sahin H, Aslantas M, Altun I, Best proximity and best periodic points for proximal nonunique contractions, J. Fixed Point Theory Appl., 23, 2021
30. Signorini A, Questioni di elasticità non linearizzata e semilinearizzata, Rendiconti di Matematica e delle Sue Applicazioni, 18(5), 1959, 95-139
31. Stampacchia G, Formes bilinéaires coercitives sur les ensembles convexes, C.R. Acad. Sci. Paris, 258, 1964, 4413-4416
32. **Stanciu M**, Best proximity for Istrăţescu mappings on generalized metric spaces, Rend. Circ. Mat. Palermo, Ser II. 74, 2025, 139
33. Suzuki T, Kikkawa M, Vetro C, The existence of best proximity points in metric spaces with the property UC, Nonlinear Anal., 71, 2009, 2918–2926
34. Tang GJ, Wang X, Wang ZB, Existence of variational quasi-hemivariational inequalities involving a set-valued operator and a nonlinear term, Optim. Lett., 9, 2015, 75-90
35. Termkaew S, Chaipunya P, Gopal D, Kumam P, A contribution to best proximity point theory and an application to partial differential equation, Optimization, 73, 2024, 2187-2219
36. Zlatanov B, Best proximity points in modular function spaces, Arabian J. Math., 4(3), 2015, 215-227