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Summary of PhD Thesis

Stochastic Perturbation of Certain Sub-Riemannian Structures

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Contents

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Given a distribution on a smooth manifold, it captures both a certain geometry as well as a specific dynamics. The geometry is given by the corresponding sub-Riemannian metric, whereas the induced dynamics is given by the horizontal (admissible) curves.

The main goal of the present Thesis is, on one hand, to investigate the geometry of rank varying sub-Riemannian structures known as Grushin-type manifolds. On the other hand, we deal with problems of connectivity (accessibility), by admissible stochastic processes, corresponding to stochastic perturbations of given distributions. These are natural objects replacing horizontal curves while passing to a stochastic framework using Wiener processes. In addition, we apply some of our methods and ideas while studying the Tzitzeica geometry of soliton solutions for certain PDEs and the Dirichlet energy on immersed tori into the hyperbolic space.

Thus, the present Thesis is naturally situated at the intersection of major fields such as: Differential Geometry, Partial Differential Equations and Probability Theory, having an interdisciplinary flavor.

Regarding the geometric aspect, we provide a detailed study of geodesics corresponding to the sub-Riemannian structure induced on $\mathbb{R}^n$ by the distribution $\mathcal{G}$, spanned locally by the vector fields $\{\partial_{x^1},\ x^1\partial_{x^2},\ x^1x^2\partial_{x^3},\ \ldots,\ x^1x^2\ldots x^{n-1}\partial_{x^n}\}$. The main result is a classification Theorem of all sub-Riemannian geodesics between two arbitrary points on the corresponding Grushin-type $n-$manifold. The classification is done with respect to the relative positions of the endpoints. Other results refer to the length of geodesics and the Carnot–Carathéodory distance, the number of geodesics joining the origin with an arbitrary point and the number of intersections of a given geodesic with the canonical submanifolds.

Concerning the problem of stochastic connectivity, our goal is to determine suitable controls which steer an admissible stochastic process, such that the probability of it reaching an arbitrarily small disk, centered at a given point, becomes close enough to one. We solve this problem for the Grushin plane endowed with a distribution of arbitrary step. Similar results are obtained for Grushin-type manifolds and for posynimial distributions.

We also introduce two classes of distributions describing the graphs of solutions to the quartic interaction PDE. We show that in both cases the Tzitzeica curvature tensor vanishes on the corresponding submanifolds.

Lastly, we show that the Dirichet energy, attached to a smooth immersion and a moving frame on tori immersed in $\mathbb{H}^n$, is bounded below by $2\pi^2$.

A sub-Riemannian manifold is a triplet $(M, \mathcal{H}, g)$ which consists of a connected, smooth, $n-$dimensional manifold M, a horizontal non-integrable distribution $\mathcal{H}$, and a Riemannian metric g defined on $\mathcal{H} \times \mathcal{H}$. One of the peculiarities of sub-Riemannian geometry, in contrast to the classical Riemannian geometry, is the fact that there can be points arbitrarily close which can be joined by more than one geodesic. Even by an infinite number of geodesics.

As a research field on its own, sub-Riemannian geometry became during the 80'ties throughout the works of Gromow [48, 49], Mitchell [65], Pansu [73, 74] and others. Before that period many problems which are now considered to be the subject of sub-Riemannian geometry were discussed in the context of *geometric control theory*, *symplectic and contact geometry*, *diffusion on manifolds*, *gauge theory* as well as *analysis of hypoelliptic operators* [47, 63, 79]. To some extent, this state of affairs has led to the situation in which sub-Riemannian geometry is also known as *Carnot-Carathéodory geometry* (Gromow [48], Pansu [73], Strichartz [81]) or *non-holonomic geometry* (Vershnik [108], Vranceanu [109, 110, 111]).

For a general introduction into the subject of sub-Riemannian geometry, we refer the reader to Bellaïche and Risler (eds.) [10], Calin [15], Calin and Chang [16], Montgomery [66].

In this Thesis we are concerned with rank varying sub-Riemannian structures known as *Grushin manifolds* [16, 114, 115], also appearing in the literature under the name of *almost-Riemannian manifolds* (see Agrachev *et al.* [1], [2], Boscain *et al.* [13]). The subject is still very actively investigated by many authors from different viewpoints.

For problems with an emphasis on heat kernels of the associated hypoelliptic operators see for instance Beals *et al.* [8], Calin *et al.* [17], Chang *et al.* [24], Chang *et al.* [25, 26].

The relationship between Grushin manifolds and the Heisenberg group, as well as isoperimetric problems on Grushin plane, are discussed by Arcozzi and Baldi in [3], by Monti and Morbidelli in [69]. For recent problems related to Grushin structures on cylinders and spheres, one might consult the recent paper by Boscain *et al.* [13].

In recent papers by Wu [114, 115] one can find new results regarding the problem of bi-Lipschitz embedding and the fractal nature of singular hyperplanes of Grushin manifolds.

The paper authored by Romney [77], motivated as well by the problem of bi-Lipschitz embedding, introduces a new conformal Grushin structure.

A comprehensive study of geodesics on the Grushin plane has been done by Calin *et al.* [17] and by Chang *et al.* [24]. Based on the information about geodesics, the authors have constructed the heat kernel for the Grushin operator

$$\Delta = \frac{1}{2}\left(\partial_x^2 + x^2\partial_y^2\right),$$

with the associated Grushin distribution $\{\partial_x, x\partial_y\}$. Some generalizations can be seen also in Chang and Li [25, 26]. The methods and ideas which had been put into practice in these works are based on earlier papers by Beals, Gaveau and Greiner, which deal with similar problems for Heisenberg manifolds [7, 8, 46].

Similar ideas have served as starting points for the first part of this Thesis which is dedicated to the study of geometry induced on $\mathbb{R}^n$ by the distribution $\mathcal{G} = \{\partial_{x^1},\ x^1\partial_{x^2},\ x^1x^2\partial_{x^3}, \dots,\ x^1x^2\dots x^{n-1}\partial_{x^n}\}$.

In a more general setting, to a given set of linearly independent vector fields $\mathcal{F} = \{X_1, \dots, X_k\}$, one associates the second order elliptic operator

$$\Delta_{\mathcal{F}} = \frac{1}{2}\sum_{i=1}^{n} X_i^2 + \dots$$

together with its heat kernel

$$p(t,x,x_0) = \left(\frac{1}{2\pi t}\right)^{n/2} e^{-\frac{d^2(x,x_0)^2}{2t}} \left(\sum_{i=0}^{\infty} a_i t^i\right).$$

Notice that here $d(x,x_0)$ denotes the distance between x and x_0 and is associated to the sub-Riemannian metric g, which turns locally the given set of vector fields into an orthonormal frame. It is worth mentioning that information about *all* geodesics is needed in order to construct the heat kernel (see for instance [7]). Finding heat kernels is sometimes quite challenging. For more details, we refer the reader to Chavel [27].

Recently, the study of *stochastically perturbed* sub-Riemannian structures has been initiated by Calin, Udrişte and Ţevy [21, 22]. Problems raised in these papers have served as starting points for some topics discussed in the Thesis, namely, *stochastic connectivity* by admissible processes and *stochastic geodesics*. Let us be more specific about what is meant by this. For simplicity we suppose that the base manifold is $\mathbb{R}^n$. Given a distribution $\mathcal{D}$, spanned locally by some given tangent vector fields $X_1, X_2, \dots, X_k$, the horizontal curves (controlled trajectories)

$$x : [0,\infty) \to \mathbb{R}^n,$$

are solutions of the ODE system

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t) X_i(x(t)). \tag{1}$$

Very often, especially in applications, one requires a model which takes into account the random disturbing effects. This technique of adding some "white noise" to a "signal", has been described, for example, by Calin *et al.* [21], Evans [37], Øksendal [72]. There are two ingredients which one uses. The first one is an $m-$dimensional Wiener process $W_t = (W_t^1, \ldots, W_t^m)^T$, $m \leq n$, whereas the second one is a matrix of functions (needed to control the amplitude) $\sigma = (\sigma_{ij})$, $1 \leq i \leq n, 1 \leq j \leq m$, $\sigma_{ij} : [0, \infty) \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$, where U is the set of admissible controls.

Having at hand these two ingredients one *stochastically perturbs* the ODE system (1), turning it into the controlled SDE (stochastic differential equation) system

$$dx_t = \left(\sum_{i=1}^{n} u_i(t) X_i(x(t)) \right) dt + \sigma dW_t. \tag{2}$$

This is a more particular form of a stochastically perturbed Itô - Pfaff system

$$dx_s = b(s, x_s, u_s) ds + \sigma(s, x_s, u_s) dW_s, \tag{3}$$

describing the problem of stochastic controlled dynamics (see for instance [34, 35, 37, 72]). Here $b : [0, \infty) \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is some given function and the control $u_s = u(s, \omega)$ is a stochastic process measurable w. r. t. the σ-algebra generated by $\{W_{s \wedge \tau}, \tau \geq 0\}$. By solutions to the above Itô–Pfaff system one means Itô processes

$$x_t = (x_1(t), x_2(t), \ldots, x_n(t)),$$

such that

$$x_t = x_0 + \int_0^t b(s, x_s, u_s) ds + \int_0^t \sigma(s, x_s, u_s) dW_s. \tag{4}$$

Thus, the basic idea is to replace horizontal curves, defined by the ODE system (1), by admissible stochastic processes defined by the SDE system (2).

Once the admissible stochastic processes are defined on a sub-Riemannian manifold, one of the first problem to consider is that of stochastic connectivity (accessibility). This would be the stochastic analogue of the Chow-Rashevskii Theorem on connectivity of sub-Riemannian manifolds (see [30, 78]). This problem, to our knowledge, has been first formulated and motivated by Calin *et al.* in [21] and has been further generalized in [86] and in [90]. In this Thesis, problems of this kind are solved for a class of Grushin-type manifolds.

It is important to point out that in the stochastic setting, the boundary conditions need to be adjusted as well. More precisely, given two arbitrary points P and Q, the probability that an admissible process starting at Q, can be steered such that it hits the point P is almost zero. Hence, it is only required that the admissible process reaches an arbitrarily small neighborhood of P.

Going one step further, one might become interested in admissible processes satisfying some optimum criteria. Such processes can be interpreted as "stochastic geodesics". There are various ways to approach this problem. Here we adopt the viewpoint from Calin *et al.* [22], which makes use of the stochastic Hamiltonian introduced by Udrişte and Damian [32, 92], solving the problem of stochastic geodesics for the Grushin plane. In this Thesis we discuss this kind of problems for an arbitrary step Grushin distribution on the plane, as well as for certain submanifolds described by solutions to the quartic interaction PDE. More precisely, our achievements in this direction are the SDE systems, describing the stochastic geodesics, analogous to the canonical equations of Hamiltonian mechanics. Solving these systems is quite challenging, although numerical methods are applicable.

One of the most famous results which naturally links the fields of Differential Geometry, Partial Differential Equations and Probability Theory is perhaps Varadhan's large deviation formula (see [107])

$$\lim_{t\to 0} -4t \log p(t,x,y) = d^2(x,y).$$

It relates the heat kernel $p(t,x,y)$ to the distance on a compact Riemannian manifold. This result has been generalized for Dirichlet spaces (for a survey, see for instance [4]) by Ariyoshi and Hino [5], Hino and Ramirez [52], showing that

$$\lim_{t\to 0} -4t \log \mathbb{P}\left(X_0 \in A; X_t \in B\right) = d^2(x,y),$$

where X_t is the Markov process attached to the regular Dirichlet form. There are three objects which come into play: a generic "Laplacian" (a second order elliptic operator), its associated heat kernel and a Wiener process, or more generally, a Markov process.

The behavior of a Wiener process (standard Brownian motion), in a probabilistic sense, is determined uniquely by the initial distribution and by the transition law from one state to another. This is the case for any Markov process (see for instance Chung [31]). There are two possible ways to specify the transition law. Namely, by its *infinitesimal generator* or by the transition *density function.*

To illustrate the above said, let us look at Brownian motion on $n-$dimensional

Euclidean space $\mathbb{R}^n$. Its infinitesimal generator, in this case, is the Laplacian

$$\Delta = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

whereas the density function is the Gaussian heat kernel

$$p(t, x, x_0) = \left(\frac{1}{2\pi t}\right)^{n/2} e^{-\frac{|x-x_0|^2}{2t}}.$$

The law of Brownian motion which starts at a given point x_0, denoted by $\mathbb{P}_{x_0}$, defines a probability measure on the space of continuous paths $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. This can be done, for example, by applying Dynkin's formula (a stochastic generalization of the second fundamental theorem of calculus),

$$\mathbb{E}_{x_0}\left[f\left(X_t\right)\right] = f(x_0) + \frac{1}{2}\int_0^t \Delta f\left(X_s\right) ds,$$

where X denotes the coordinate process on the space of continuous paths, *i.e.*,

$$X(\omega)_t = X_t(\omega) = \omega_t, \quad \omega \in \mathcal{C}\left(\mathbb{R}_+, \mathbb{R}^n\right),$$

and $\mathbb{E}_{x_0}$ is the expected value. Thus, Dynkin's formula defines uniquely the measure $\mathbb{P}_x$ such that the coordinate process has the Gaussian density function and is a Markov process. The paths of Brownian motion can be thought as the characteristic lines of the Laplacian Δ.

When passing to Riemannian manifolds, the role of the Laplacian, as the infinitesimal generator of Brownian motion, is taken over by the Laplace-Beltrami operator. Quite a lot is known about Brownian motion on Riemannian manifolds (see for instance [56, 57, 82]), meanwhile a similar theory for sub-Riemannian manifolds is still developing. This Thesis aims to contribute in this direction as well.

Original contribution

In Chapter 1 of the present Thesis, entitled **The geometry of Grushin-type manifolds**, we investigate the geometry induced by the distribution

$$\mathcal{G} = \{\partial_{x^1}, x^1\partial_{x^2}, x^1x^2\partial_{x^3}, \dots, x^1x^2\dots x^{n-1}\partial_{x^n}\},$$

on the real $n-$dimensional manifold $\mathbb{R}^n$.

The sub-Riemannian metric $g = (g_{ij})$, corresponding to the distribution $\mathcal{G}$, is given by $g_{11} = 1$, $g_{ij} = \delta_{ij}\left(x^1 \dots x^{i-1}\right)^{-2}$, $i = 2, \dots, n$, where δ_{ij} is Kroneker's delta, and is defined only outside the hyperplanes $\{x^i = 0\}$. The resulting Grushin manifold in this case is the triplet $\mathbb{G}_n = (\mathbb{R}^n, \mathcal{G}, g)$.

The original contribution of Chapter 1 is as follows. Theorem 1.4.3 describes the case when there is a unique geodesic joining two arbitrarily given points. Theorem 1.4.4 provides the lengths of geodesics giving the Carnot-Carathéodory-Vrănceanu distance. The main result of the Chapter is Theorem 1.5.4, which gives a complete classification of sub-Riemannian geodesics. More precisely, we establish the conditions under which the number of geodesics between two arbitrary points is one, countably infinite, and finite respectively. Theorem 1.6.1 provides the number of intersection points of an arbitrary geodesic with the canonical submanifolds and Theorem 1.6.3 establishes the number of geodesics joining the origin with an arbitrary point. Lemma 1.5.3 and Lemma 1.6.2, respectively, are important technical results.

The original results, for a three dimensional case, are published in [89] (T. Ţurcanu, *On sub-Riemannian geodesics associated to a Grushin operator*, Appl. Anal., ID: 1268685 (IF 0.815)). These results add up naturally to ones obtained previously by Beals *et al.* [8], Calin *et al.* [17, 19], Chang *et al.* [24], Chang *et al.* [25, 26].

The distribution $\mathcal{G}$ being bracket generating, the global connectivity by horizontal curves is guaranteed. In other words the Carnot-Carathéodory-Vrănceanu distance on $\mathbb{G}_n$ is finite. The next step is to look for curves minimizing this distance (called henceforth geodesics). These are obtained by projecting, onto the base space $\mathbb{R}^n$, the bicharacteristic curves of the Hamiltonian function

$$H(x,p) = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}p_ip_j = \frac{1}{2}\left(p_1^2 + \left(x^1\right)^2 p_2^2 + \dots + \left(x^1 \dots x^{n-1}\right)^2 p_n^2\right),$$

defined on the cotangent bundle $T^*\mathbb{R}^n$. The solutions of the canonical Hamiltonian system provide the equations of the geodesics. These are

$$
\begin{aligned}
x^n(t) &= \frac{1}{4}\left(\frac{C_{n-1}}{C_n}\right)^2 \left[2\varphi_{n-1}(t) - \sin\left(2\varphi_{n-1}(t)\right) - \left(2\varphi_{n-1}^0 - \sin\left(2\varphi_{n-1}^0\right)\right)\right], \\
\varphi_i(t) &= \frac{1}{4}\frac{C_{i+1}C_{i-1}^2}{C_i^3}\left[2\varphi_{i-1}(t) - \sin\left(2\varphi_{i-1}(t)\right) - \left(2\varphi_{i-1}^0 - \sin\left(2\varphi_{i-1}^0\right)\right)\right] + \varphi_i^0, \\
x^i(t) &= \frac{C_i}{C_{i+1}}\sin\left(\varphi_i(t)\right), \\
p_i(t) &= C_i\cos\left(\varphi_i(t)\right), \quad i = 2, \ldots, n-1,
\end{aligned}
$$

where $x^1(t) = C_1 \sin\left(C_2 t + \alpha_1\right)$, $\varphi_1(t) = C_2 t + \alpha_1$, and C_i, φ_i^0 are constants.

The case when there is only one geodesic connecting two arbitrary points is described in the following (the numbering appearing in the brackets is the same as that of the main text)

Theorem (1.4.3)**.** *If $C_2 = 0$ and $x_0^k \neq 0$, $k = 2, \ldots, n$, then $x_0^k = x_1^k$ and there exists a unique geodesic*

$$x : [0,1] \longrightarrow \mathbb{R}^n, \quad x(t) = \left(\left(x_1^1 - x_0^1\right)t + x_0^1,\ x_0^2, \ldots, x_0^n\right),$$

connecting the points $P(x_0)$ and $Q(x_1)$, of length

$$\ell\left[x(t)\right] = |x_1^1 - x_0^1|.$$

The length of an arbitrary geodesics, which is used to compute the Carnot-Carathéodory-Vrănceanu distance, is given by the next

Theorem (1.4.4)**.** *With the above notation and definitions consider $C_2 > 0$. Then, the length of a geodesic $x(t)$ is*

$$\ell\left[x(t)\right] = C_2|C_1|.$$

The main result of the first Chapter, which classifies the geodesics according to the position of the endpoints, is as follows.

Theorem (1.5.4)**.** *Let $P(x_0)$ and $Q(x_1)$ be two given points in the Grushin space $\mathbb{G}_n$. The number of geodesics joining them is*

i) one, if $x_0^i = x_1^i \neq 0$, $\forall i = 2, \ldots, n$;

ii) countably infinite, if there exists $i \in \{1, \ldots, n-1\}$ such that $x_0^i = x_1^i = 0$;

iii) finite, if otherwise.

The number of intersection points of an arbitrary geodesic with the canonical manifold, for the three dimensional case, is determined by the following

Theorem (1.6.1). *Let $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ be two points in $\mathbb{G}_3$ and let $x(t)$ be a geodesic joining them. The number n, of intersection points of the given geodesic*

i) with the plane yOz, is

$$n = \begin{cases} \left[\dfrac{C_2}{\pi}\right] + 1, & \text{for } \alpha = 0 \\ \\ \left[\dfrac{C_2 + \alpha}{\pi}\right] - \left[\dfrac{\alpha}{\pi}\right], & \text{for } \alpha \in (-\pi, \pi) \setminus \{0\}; \end{cases}$$

ii) with the plane xOz, is

$$n = \begin{cases} \left[\dfrac{\varphi_1}{\pi}\right] + 1, & \text{for } \varphi_0 = 0 \\ \\ \left[\dfrac{\varphi_1}{\pi}\right] - \left[\dfrac{\varphi_0}{\pi}\right], & \text{for } \varphi \in (-\pi, \pi) \setminus \{0\}; \end{cases}$$

iii) with the Oz axis, is $|\Gamma_\psi \cap \mathbb{T}|$, where, respectively,

$$\Gamma_\psi = \{(t, \psi(t)) \in \mathbb{R}^2 | \ \psi(t) = \frac{1}{2} p_3 C_1^2 t + \frac{1}{4} p_3 C_1^2 \sin 2\alpha + \varphi_0\},$$

$$\mathbb{T} = \{(l\pi, m\pi) \in \mathbb{R}^2 | \ 0 \leq l \leq \left[\frac{C_2 + \alpha}{\pi}\right], \ 0 \leq m \leq \left[\frac{\varphi_1}{\pi}\right]\}.$$

The last theorem of the Chapter, which provides the number of geodesics starting at the origin and reaching a point outside the canonical manifolds, is stated as follows.

Theorem (1.6.3). *With the above notation and definitions, let the point P be at the origin and let $Q(x_1, y_1, z_1)$ be a point such that $x_1 y_1 \neq 0$, and denote by $\varphi_1, \ldots, \varphi_n$ the solutions of the equation $\mu(\varphi) = \dfrac{2z_1}{y_1^2}$. Then,*

i) the number n is given by

$$n = 2\left[\frac{2z_1}{\pi y_1^2}\right] + sgn\left(\frac{2z_1}{y_1^2} - \pi\left[\frac{2z_1}{\pi y_1^2}\right] - \arctan\left(\frac{2z_1}{y_1^2}\right)\right),$$

ii) the number of geodesics between P and Q is $N = m_1 + \cdots + m_n$, where

$$m_i = 2\left[\frac{2\varphi_i y_1}{\pi x_1^2 \sin\varphi_i}\right] + sgn\left(\frac{2\varphi_i y_1}{x_1^2 \sin\varphi_i} - \pi\left[\frac{2\varphi_i y_1}{\pi x_1^2 \sin\varphi_i}\right] - \arctan\left(\frac{2\varphi_i y_1}{x_1^2 \sin\varphi_i}\right)\right), \; i = 1, \ldots, n.$$

In the second Chapter, entitled **Stochastic connectivity on a perturbed Grushin distribution**, we prove a stochastic connectivity property for a stochastically perturbed step $k+1$ Grushin distribution. The main result of the Chapter is Theorem 2.3.2, which provides suitable control functions steering any admissible processes, starting at a point P, such that the probability of it striking any arbitrarily small disk centered at a point Q, becomes close enough to one. Corollary 2.4.1 extends the main result to the case when both endpoint conditions are expressed in probabilistic terms.

The original results of this Chapter are published in [86] (T. Ţurcanu, C. Udrişte, *Stochastic perturbation and connectivity based on Grushin distribution*, U. Politeh. Bucharest Sci. Bull. Ser. A, 79, 1 (2017), 3-10 (IF 0.365)), naturally extending the results of Calin, Udrişte and Ţevy [21].

The first (geometric) ingredient is the bracket generating, step $k+1$, Grushin distribution $\{\partial_x, x^k\partial_y\}$, $k \in \mathbb{N}^*$, defined on the real plane. The second (stochastic) ingredient, is a 2−dimensional Wiener process (standard Brownian motion).

Let $\mathcal{U}_1$ denote the set of *deterministic (open loop)* controls, *i.e.*, controls $u(s,\omega) = u(s)$ not depending on ω, and let $\mathcal{U}_2$ denote the set of *Markov controls*, *i.e.*, functions $u(s,\omega) = u_0(s, x_s(\omega))$, such that $u_0 : \mathbb{R}^{n+1} \to U \subset \mathbb{R}^k$. Following Calin *et. al* [21], a stochastic process $c_s = (x(s), y(s))$, which satisfies the SDE system

$$\begin{cases} dx(s) = u_1(s)ds + \sigma_1 dW_s^1 \\ dy(s) = u_2(s)x^k(s)ds + \sigma_2 dW_s^2, \end{cases}$$

with $u_1, u_2 \in \mathcal{U}_1 \cup \mathcal{U}_2$, will be called *admissible stochastic process.*

In order to define the connectivity in the stochastic setting we have to make the above mentioned adjustments. Given a stochastic process X_t, starting at a given initial point P, it is clear that the probability of X_t striking another point Q at $T > 0$, is zero. Therefore, we have to leave the endpoint X_T free and consider an arbitrarily small disk centered at Q.

The stochastic connectivity property, with respect to the stochastically perturbed Grushin distribution, is established by the following

Theorem (2.3.2). *Let $P=(x_P,y_P), Q=(x_Q,y_Q)$ be two points in $\{\mathbb{R}^2,\mathcal{D}_G\}$ and denote by $D(Q,r)$ the Euclidean disk of radius r, centered at Q. Then, for any $\varepsilon\in(0,1)$, and $r>0$, there exists a striking time $t<\infty$ and an admissible stochastic process c_s, satisfying the boundary conditions*

$$(x(0),y(0))=(x_P,y_P)\,,\ \ (\mathbb{E}\left[x(t)\right],\mathbb{E}\left[y(t)\right])=(x_Q,y_Q)\,,$$

such that

$$\mathbb{P}\left(c_t\in D(Q,r)\right)\geq 1-\varepsilon.$$

As one can notice, in the above Theorem, the endpoints of an admissible stochastic process do not play exactly the same role. More precisely, the initial condition is given in deterministic terms, whereas the final configuration is expressed in probabilistic terms. The main result can be restated such that the endpoints become interchangeable.

Theorem (2.4.1). *Let P,Q be two arbitrary points in $\{\mathbb{R}^2,\mathcal{D}\}$. Then, for any $r_1,r_2>0$, $0<\varepsilon_1,\varepsilon_2<1$, there exist t_1 and t_2, respectively, and an admissible stochastic process c_s, satisfying the boundary conditions*

$$(\mathbb{E}\left[x(t_1)\right],\mathbb{E}\left[y(t_1)\right])=(x_P,y_P)\,,\ \ (\mathbb{E}\left[x(t_2)\right],\mathbb{E}\left[y(t_2)\right])=(x_Q,y_Q)\,,$$

such that

$$\mathbb{P}\left(c_{t_1}\in D(P,r_1)\right)\geq 1-\varepsilon_1,\ \ \mathbb{P}\left(c_{t_2}\in D(Q,r_2)\right)\geq 1-\varepsilon_2.$$

The last Section of Chapter 2 addresses the problem of stochastic geodesics on the corresponding Grushin $2-$manifold. We obtain the SDE system describing the geodesics and, surprisingly, find out that there are no deterministic controls which can solve the problem (Proposition 2.5.2).

In Chapter 3, entitled **Stochastic accessibility along a perturbed posynomial distribution**, we continue the study of accessibility problems associated to sub-Riemannian structures. This time a much wider class of distributions is considered, namely, posynomial distributions.

The main result of the Chapter is Theorem 3.3.2, which is an accessibility (connectivity) result analogue to Theorem 2.3.2.

The main result together with Lemma 3.3.1, are published in [90] (T. Ţurcanu, C. Udrişte, *Stochastic accessibility on Grushin-type manifolds*, Statist. Probab. Lett., 125 (2017), 196-201 (IF 0.506)). Let us mention that in [90] the setting is slightly modified. The base manifold is $\mathbb{R}^n$, whereas the distribution has integer exponents.

Our main geometric ingredients are as follows. The base manifold is the space of strictly positive $n-$tuples $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) | \ x_i > 0, \ i = 1, \ldots, n\}$, on which the posynomial distribution $\mathcal{P}$ is defined locally as the span of the vector fields

$$\begin{array}{lll} X_1 = & \mu^1(x)\partial_{x_1} := & \partial_{x_1} \\ X_2 = & \mu^2(x)\partial_{x_2} := & x_1^{k_1}\partial_{x_2} \\ X_3 = & \mu^3(x)\partial_{x_3} := & x_1^{k_1}x_2^{k_2}\partial_{x_3} \\ \vdots & \vdots & \vdots \\ X_n = & \mu^n(x)\partial_{x_n} := & x_1^{k_1}x_2^{k_2}\ldots x_{n-1}^{k_{n-1}}\partial_{x_n}. \end{array}$$

The corresponding sub-Riemannian metric g, in this case, is defined as $g_{11} = 1$, $g_{ij} = \delta_{ij}\left(x_1^{k_1}\ldots x_{i-1}^{k_{i-1}}\right)^{-2}$, $i = 2, \ldots, n$. Nevertheless, it is not used in Chapter 3 as its corresponding topology is equivalent to that of the Euclidean metric (see for instance [43]). Hence, for our purposes, instead of Carnot-Carathéodory-Vrănceanu disks we may use Euclidean disks. Recall that the problem of accessibility in a stochastic framework requires the endpoint of a process to be left free.

As before, putting together the posynomial distribution with an $n-$dimensional Wiener process $(W_s^1, \ldots, W_s^n)$, yields a stochastically perturbed Pfaff system. The admissible stochastic processes are defined correspondingly. The main result of Chapter 3, reads as follows.

Theorem (3.3.2)**.** *Consider two arbitrary points in $\mathbb{R}^n_+$, denoted by $P = (x_1^P, \ldots, x_n^P)$ and $Q = (x_1^Q, \ldots, x_n^Q)$ respectively, and let $D(Q, r)$ denote the Euclidean disk of radius r centered at Q. Then, for any fixed $\varepsilon \in (0, 1)$ and $r > 0$, there exists a time $t < \infty$ and an admissible stochastic process x_s, such that*

$$\mathbb{P}\left(x_t \in D(Q, r)\right) \geq 1 - \varepsilon,$$

and which satisfies the boundary conditions

$$x_0 = P, \quad \mathbb{E}\left[x_t\right] = Q.$$

In Chapter 4, entitled **The geometry of solutions for quartic interaction PDE**, we study the interplay between PDEs and Differential Geometry from another perspective and we discuss some questions in a stochastic setting as well. The main ingredient this time is a completely integrable smooth distribution, a semi-Riemannian base manifold and a d'Alembertian.

We are interested in the geometric properties of graphs of functions arising as solutions for the quartic interaction PDE. We discuss two classes of solutions. The first one is the class represented by soliton solutions, whereas the second class consists of solutions of a certain first order PDE system, which generates the quartic interaction PDE, in the sense of least squares action.

The main results are Theorem 4.3.1 and Theorem 4.4.3, which show that in both cases the graphs of solutions are Tzitzeica flat, *i.e.*, the associated curvature tensor vanishes. We also have Theorem 4.4.2 which shows that the quartic interaction PDE can be generated using a least squares type action. At the end of the chapter we introduce the notion of stochastic geodesics on graphs of soliton solutions and obtain the SDE system describing them.

The original results of Chapter 4 are published in [87] (T. Ţurcanu, C. Udrişte, *Tzitzeica geometry of soliton solutions for quartic interaction PDE*, Balkan J. Geom. Appl., 21, 1 (2016), 103-112).

The Klein-Gordon equation is a fundamental equation of the Quantum Field Theory. It can be altered in such a manner that the solutions of the modified version, which is called *quartic interaction PDE*, are fields with quartic interaction (see for example [76]). The quartic interaction PDE, defined on the four dimensional Minkowski space-time , writes as

$$\Box u := u_{11} - u_{22} - u_{33} - u_{44} = \mu^2 u - \lambda u^3,$$

where μ is the *mass term*, λ is the (strictly positive) *coupling constant*, and $\Box$ is the d' Alembert operator (with $c = 1$). The graphs of soliton solutions are at the same time integral manifolds of the distribution $\mathcal{D}$, spanned locally by the vector fields

$$\begin{aligned} Y_1 &= (1,0,0,0,k_1 Y(u)), \quad Y_2 = (0,1,0,0,k_2 Y(u)), \\ Y_3 &= (0,0,1,0,k_3 Y(u)), \quad Y_4 = (0,0,0,1,k_4 Y(u)), \end{aligned}$$

where k_1, k_2, k_3, k_4 are some constants and $Y(u)$ is a certain function of u.

The first important result of Chapter 4 is the following

Theorem (4.3.1)**.** *Let S be an integral manifold of the distribution $\mathcal{D}$. Then*

i) the components of the Tzitzeica connection are

$$\Lambda^{\gamma}_{\alpha\beta} = h^{\gamma\sigma} Y^5_\sigma \frac{\partial Y^5_\alpha}{\partial u} Y^5_\beta = (h^{\gamma\sigma} k_\sigma)\, k_\alpha k_\beta\, (Y)^2 \frac{\partial Y}{\partial u},$$

ii) the curvature tensor of (S, Λ) is identically zero.

The next result shows that a certain first order normal PDE system is a generator, in the sense of least squares action, of the quartic interaction PDE. Namely,

Theorem (4.4.2)**.** *i) The quartic interaction PDE is an Euler-Lagrange prolongation of the system*

$$\begin{cases} \dfrac{\partial x^i}{\partial t^\alpha} = \delta^i_\alpha = X^i_\alpha(x(t)), \quad i, \alpha = 1, 2, 3, 4, \\ \\ \dfrac{\partial x^5}{\partial t^\alpha} = X^5_\alpha(x(t)). \end{cases}$$

ii) There exist infinitely many suitable geometric structures and infinitely many vector fields which realize the above prolongation.

The relevance of the above system relies on the fact that its solutions solve the quartic interaction PDE as well (for more about this approach see [95]-[105]). The distribution $\mathcal{D}'$, associated to this system, has precisely the same form as $\mathcal{D}$, with a certain function $X(u)$ replacing the function $Y(u)$. The statement of the Theorem 4.3.1 is true for the distribution $\mathcal{D}'$ as well.

In Chapter 5, entitled **Dirichlet frame energy on a torus immersed in** $\mathbb{H}^n$ we study the boundedness of the Dirichlet energy attached to moving frames on a torus immersed a hyperbolic space. We also introduce a stochastic version of the Dirichlet energy.

The main result of Chapter 5 is Theorem 5.2.1, together with Corollary 5.2.2, showing that the Dirichlet energy associated to a pair consisting of an immersion and a moving frame, is bounded below strictly by $2\pi^2$. Analogous bounds were obtained by Mondino *et al.* for immersions into $\mathbb{R}^n$ [64] and Topping [83] for the $n-$Sphere.

The original results of this Chapter are published in [88] (T. Ţurcanu, C. Udrişte, *A lower bound for the Dirichlet energy of moving frames on a torus immersed in* $\mathbb{H}^n$, Balkan J. Geom. Appl., 20, 2 (2015), 84-91).

The starting point of our study in Chapter 5 is an abstract torus denoted by $\mathbb{T}$, a smooth immersion $\varphi : \mathbb{T} \hookrightarrow \mathbb{H}^n, n \geq 3$ and a moving frame on $\varphi(\mathbb{T})$, which is a pair of sections in the tangent bundle $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$.

The pullback metric, denoted by $h := \varphi^* g_{\mathbb{H}^n}$, is induced naturally by the immersion φ. The Dirichlet energy associated to the pair $(\varphi, \mathbf{x})$, is the functional

$$\mathcal{D}(\varphi, \mathbf{x}) = \frac{1}{4} \int_{\mathbb{T}} |d\mathbf{x}|^2 d\mu_h, \tag{5}$$

with d being the differential of the frame.

Following Mondino *et al.* [64], we reduce the problem to the case of coordinate moving frames associated to smooth immersions of flat tori. Our main result is the following

Theorem (5.2.1)**.** *Consider a smooth conformal immersion* $\varphi : \Sigma \hookrightarrow \mathbb{H}^n, n \geq 3$ *and let* $\mathbf{x}$ *be the attached coordinate moving frame. Then the following inequality holds true*

$$\mathcal{D}(\varphi, \mathbf{x}) = \frac{1}{4} \int_\Sigma |d\mathbf{x}|^2 d\mu_h > \pi^2 \left(b + \frac{1}{b} \right) \frac{1}{1 + \cot^2 \theta \cos^2 \theta}. \tag{6}$$

As a corollary, we obtain that the Dirichlet energy is bounded below by $2\pi^2$.

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