Results in Fixed Point Theory and Iteration Processes with Applications

Asst. drd. Adrian Sorinel Ghiura
Department of Mathematics & Informatics
University "Politehnica" of Bucharest

PhD Thesis Abstract

Supervised by Prof. Dr. habil. Mihai Postolache

Bucharest, May, 2017

Contents

Keywords

Caristi's theorem, C^*-algebra, metric space, C^*-algebra-valued metric, b-metric space, contractive mapping, fixed point theorem, asymptotically nonexpansive self and non-self mapping in intermediate sense, new two-step iteration scheme of mixed type, common fixed point, uniformly convex Banach space, strong convergence, weak convergence, fixed point, rate of convergence, quasi variational inequalities, projection operator, iterative methods, convergence, Wiener-Hopf equations.

Author's published papers

1. Shehwar, D, Batul, S, Kamran, T, **Ghiura, A**: *Caristi's fixed point theorem on C^*-algebra-valued metric spaces.* J. Nonlinear Sci. Appl. **9**, 584-588 (2016)

2. Saluja, GS, Postolache, M, **Ghiura, A**: *Convergence theorems for mixed type asymptotically nonexpansive mappings in the intermediate sense.* J. Nonlinear Sci. Appl. **9**, 5119-5135 (2016)

3. Noor, MA, Noor, KI, Khan, AG, **Ghiura, A**: *Iterative algorithms for solving a class of quasi variational inequalities.* U.P.B. Sci. Bull. Ser. A **78**(3), 3-18 (2016).

4. Kamran, T, Postolache, M, **Ghiura, A**, Batul, S, Ali, R: *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application.* Fixed Point Theory Appl. 2016:10 (2016)

5. Fathollahi, S, **Ghiura, A**, Postolache, M, Rezapour, S: *A comparative study on the convergence rate of some iteration methods involving contractive mappings.* Fixed Point Theory Appl. 2015:234 (2015)

Deeply indebted to my advisors

I use this opportunity to express my appreciation and my sincere gratitude for the helpful guidance provided by the following scientists:

Prof. Dr. Tayyab Kamran, Quaid-i-Azam University, Islamabad and M.U. Ali, National University of Computer and Emerging Sciences, Islamabad, for our joint research on

C^*-algebra-valued metric space.

Dr. Gurucharan Saluja, Govt. Nagarjuna P.G. College of Science, Raipur, which guided my step-by-step research on Iteration Theory, the core of Chapter 2.

Prof. Dr. Shahram Rezapour, Azarbaijan Sahid Madani University, for the kindness to accept me in his research group. I acknowledge the constructive discussion during his visit to our department, and I am grateful to him for the comparative study of some iteration processes.

Prof. Dr. Muhammad Aslam Noor, COMSATS, a leading expert in Nonlinear Analysis. Under his valuable guidance I realized the study on iterative algorithms for variational inequalities and the findings in this direction, the core of Chapter 4.

Prof. Dr. hab. Mihai Postolache, University Politehnica of Bucharest, my supervisor, for his patience to accept anytime professional discussions on the subject of this Thesis.

PhD Abstract

In this PhD Thesis, we present our contribution to Fixed Point Theory and some applications to Iteration Processes and Variational Inequalities. The study is motivated by nowadays research developed by leading scientists and by its possible development for real world applications; please, see: Bakhtin [7], Banach [8], Berinde [13], Czerwick [19], Mann [38], Noor *et al.* [45], Saluja [52], Thakur *et al.* [60]. The results are published in selective journals such as: J. Nonlinear Sci. Appl., Fixed Point Theory Appl., and U.P.B. Sci. Bull. Ser. A.

In Chapter 1, titled **Fixed point results in C^*-algebra-valued metric spaces**, we present our fixed point results in the framework of C^*-algebra-valued metric spaces. Based on the concept and the properties of C^*-algebras, we present an extension of Caristi's fixed point theorem for mappings defined on C^*-algebra-valued metric spaces. Also, we introduce the notion of a C^*-algebra-valued b-metric space and generalize the Banach contraction principle in this new setting. The study in this chapter should be thought as a natural continuation of those of: Batul and Kamran [9], Khamsi and Kirk [30], Czerwick [18], Ma *et al.* [37].

The original contribution in this chapter is: Definition 1.5, Example 1.1, Lemma 1.1, Theorem 1.1, Theorem 1.2, Example 1.2, Definition 1.7, Example 1.3, Definition 1.8, Example 1.4, Theorem 1.3, Example 1.5, Application.

They are published in [55] and [29] (Shehwar, D, Batul, S, Kamran, T, Ghiura, A: *Caristi's fixed point theorem on C^*-algebra-valued metric spaces*, J. Nonlinear Sci. Appl. 9, 584-588 (2016) and Kamran, T, Postolache, M, Ghiura, A, Batul, S, Ali, R: *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application*, Fixed Point Theory Appl. 2016:10 (2016)).

Definition 0.1 ([37]). Let X be a non-empty set. A C^*-algebra-valued metric on X is a mapping $d\colon X \times X \to \mathbb{A}_+$ satisfying the following conditions:

(i) $0_{\mathbb{A}} \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0_{\mathbb{A}} \Leftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$ $\forall x, y \in X$,

(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ $\forall x, y, z \in X$.

The triple $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

We begin the chapter by introducing the notion of lower semi continuity in the context of C^*-algebra valued metric spaces.

Definition 0.2. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $\phi\colon X \to \mathbb{A}$ is said to be lower semi-continuous at x_0 with respect to $\mathbb{A}$ if

$$\|\phi(x_0)\| \leq \lim_{x\to x_0} \inf \|\phi(x)\|$$

Lemma 0.1. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space and let $\phi\colon X \to \mathbb{A}_+$ be a map. Define the order $\preceq_\phi$ on X by*

$$x \preceq_\phi y \Longleftrightarrow d(x, y) \preceq \phi(y) - \phi(x) \text{ for any } x, y \in X. \tag{1}$$

Then $\preceq_\phi$ is a partial order on X.

Theorem 0.1. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi-continuous map. Then $(X, \preceq_\phi)$ has a minimal element, where $\preceq_\phi$ is defined by* (1).

As a consequence of the above theorem we have the following fixed point result.

Theorem 0.2. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi continuous map. Let $T\colon X \to X$ be such that for all $x \in X$*

$$d(x, Tx) \preceq \phi(x) - \phi(Tx).$$

Then T has at least one fixed point.

In the second part, we extend the definition of a b-metric to introduce the notion b-metric space in the setting of C^*-algebras as follows.

Definition 0.3 ([37])**.** Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T\colon X \to X$ is said to be a C^*-valued contraction mapping on X if there exists $a \in \mathbb{A}$, with $\|a\| < 1$, such that

$$d(Tx, Ty) \preceq a^* d(x, y) a, \quad \text{for all } x, y \in X.$$

Definition 0.4. Let $\mathbb{A}$ be a C^*-algebra, and X be a nonempty set. Let $b \in \mathbb{A}$ be such that $\|b\| \geq 1$. A mapping $d_b\colon X \times X \to \mathbb{A}_+$ is said to be a C^*-algebra-valued b-metric on X if the following conditions hold for all $x_1, x_2, x_3 \in \mathbb{A}$:

(BM1) $d_b(x_1, x_2) = 0_{\mathbb{A}} \Leftrightarrow x_1 = x_2$;

(BM2) d_b is symmetric, that is, $d_b(x_1, x_2) = d_b(x_2, x_1)$;

(BM3) $d_b(x_1, x_2) \preceq b\,[d_b(x_1, x_3) + d_b(x_3, x_2)]$.

The triple $(X, \mathbb{A}, d_b)$ is called a C^*-algebra-valued b-metric space with coefficient b.

Definition 0.5. Let $(X, \mathbb{A}, d_b)$ be a C^*-valued b-metric space. A contraction on X is a mapping $T\colon X \to X$ if there exists $a \in \mathbb{A}$, with $\|a\| < 1$, such that

$$d_b(Tx, Ty) \preceq a^* d_b(x, y) a \quad \text{for all } x, y \in X.$$

Theorem 0.3. *Consider a complete C^*-valued b-metric space $(X, \mathbb{A}, d_b)$ with coefficient b. Let $T\colon X \to X$ be a contraction with the contraction constant a, such that $\|b\|\|a\|^2 < 1$. Then T has a unique fixed point in X.*

For examples and applications illustrating our results, please see [29, 55].

In Chapter 2, **Iterating mixed type asymptotically nonexpansive mappings**, we provide a new two-step iteration scheme of mixed type for two asymptotically nonexpansive self mappings in the intermediate sense and two asymptotically nonexpansive non-self mappings in the intermediate sense and establish some strong and weak convergence theorems for the mentioned scheme and mappings in uniformly convex Banach spaces. Our results extend corresponding results of Chidume *et al.* [15, 16], Guo *et al.* [26, 27], Saluja [52], Schu [54], Tan and Xu [59], Wang [61], Wei and Guo [62, 63].

Our original contribution in this chapter is: Example 2.1, Example 2.2, Lemma 2.5, Lemma 2.6, Theorem 2.1, Theorem 2.2, Theorem 2.3, Lemma 2.7, Lemma 2.8, Theorem 2.4, Theorem 2.5, Theorem 2.6, Example 2.3, Example 2.4, Example 2.5.

They are published in [53] (Saluja, GS, Postolache, M, Ghiura, A: *Convergence theorems for mixed type asymptotically nonexpansive mappings in the intermediate sense.* J. Nonlinear Sci. Appl. 9, 5119-5135 (2016)).

Definition 0.6. Let K be a nonempty subset of a real Banach space E and $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is said to be asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| - \|x - y\| \Big) \leq 0.$$

Wei and Guo [63] defined the new iteration scheme of mixed type with mean errors as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n), \\ y_n &= P(\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n),\ n \geq 1, \end{aligned} \tag{2}$$

where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$, and prove some weak convergence theorems in the setting of real uniformly convex Banach spaces.

The purpose of this chapter is to study the iteration scheme (2) for mixed type asymptotically nonexpansive mappings in the intermediate sense which is more general than the class of asymptotically nonexpansive mappings in uniformly convex Banach spaces, and establish some strong and weak convergence theorems for the mentioned scheme and mappings.

Lemma 0.2. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings in the intermediate sense and $T_1, T_2\colon K \to E$ two asymptotically nonexpansive non-self mappings in the intermediate sense. Put*

$$\begin{aligned} G_n &= \max\Big\{0, \sup_{x,\,y\in K,\; n\geq 1} \Big(\|S_1^n x - S_1^n y\| - \|x-y\|\Big), \\ &\qquad \sup_{x,\,y\in K,\; n\geq 1} \Big(\|S_2^n x - S_2^n y\| - \|x-y\|\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} H_n &= \max\Big\{0, \sup_{x,\,y\in K,\; n\geq 1} \Big(\|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| - \|x-y\|\Big), \\ &\qquad \sup_{x,\,y\in K,\; n\geq 1} \Big(\|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\Big)\Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Let $\{x_n\}$ be the sequence defined by (2), where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\gamma'_n < \infty$. Assume that $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \neq \emptyset$. Then $\lim_{n\to\infty}\|x_n - q\|$ and $\lim_{n\to\infty} d(x_n, F)$ both exist for any $q \in F$.

Lemma 0.3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings in the intermediate sense and $T_1, T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense and G_n and H_n be taken as in Lemma 0.2. Assume that $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2), where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\gamma'_n < \infty$. If the following conditions hold:*

(i) $\{\beta_n\}$ and $\{\beta'_n\}$ are real sequences in $[\rho, 1-\rho]$ for all $n \geq 1$ and for some $\rho \in (0,1)$.

(ii) $\|x - T_i(PT_i)^{n-1}y\| \leq \|S_i^n x - T_i(PT_i)^{n-1}y\|$ for all $x, y \in K$ and $i = 1, 2$.

Then $\lim_{n\to\infty}\|x_n - S_i x_n\| = \lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for $i = 1, 2$.

Theorem 0.4. *Under the assumptions of Lemma* 0.3*, if one of* S_1*,* S_2*,* T_1 *and* T_2 *is completely continuous, then the sequence* $\{x_n\}$ *defined by* (2) *converges strongly to a common fixed point of the mappings* S_1*,* S_2*,* T_1 *and* T_2*.*

For our next result, we need the following definition.

A mapping $T\colon K \to K$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $x_{n_r} \to x^* \in K$ strongly as $r \to \infty$.

Theorem 0.5. *Under the assumptions of Lemma* 0.3*, if one of* S_1*,* S_2*,* T_1 *and* T_2 *is semi-compact, then the sequence* $\{x_n\}$ *defined by* (2) *converges strongly to a common fixed point of the mappings* S_1*,* S_2*,* T_1 *and* T_2*.*

Theorem 0.6. *Under the assumptions of Lemma* 0.3*, if there exists a continuous function* $f\colon [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(t) > 0$ *for all* $t \in (0,\infty)$ *such that*

$$f(d(x,F)) \leq a_1 \|x - S_1x\| + a_2 \|x - S_2x\| + a_3 \|x - T_1x\| + a_4 \|x - T_2x\|$$

for all $x \in K$*, where* $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$ *and* a_1*,* a_2*,* a_3*,* a_4 *are nonnegative real numbers such that* $a_1 + a_2 + a_3 + a_4 = 1$*, then the sequence* $\{x_n\}$ *defined by* (2) *converges strongly to a common fixed point of the mappings* S_1*,* S_2*,* T_1 *and* T_2*.*

In the following, we prove some weak convergence theorems of the iteration scheme (2) for mixed type asymptotically nonexpansive mappings in the intermediate sense in real uniformly convex Banach spaces.

Lemma 0.4. *Under the assumptions of Lemma* 0.2*, for all* p_1*,* $p_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$*, the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$$

exists for all $t \in [0,1]$*, where* $\{x_n\}$ *is the sequence defined by* (2)*.*

Lemma 0.5. *Under the assumptions of Lemma* 0.2*, if* E *has a Fréchet differentiable norm, then for all* p_1*,* $p_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$*, the limit*

$$\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle$$

exists, where $\{x_n\}$ *is the sequence defined by* (2)*, if* $\omega_w(x_n)$ *denotes the set of all weak subsequential limits of* $\{x_n\}$*, then* $\langle l_1 - l_2, J(p_1 - p_2)\rangle = 0$ *for all* p_1*,* $p_2 \in F$ *and* l_1*,* $l_2 \in W_w(\{x_n\})$*.*

Theorem 0.7. *Under the assumptions of Lemma* 0.3*, if E has Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by* (2) *converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.*

Theorem 0.8. *Under the assumptions of Lemma* 0.3*, if the dual space E^* of E has the Kadec-Klee (KK) property and the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (2) *converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.*

Theorem 0.9. *Under the assumptions of Lemma* 0.3*, if E satisfies Opial's condition and the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (2) *converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.*

For examples illustrating these results, please see [53].

In Chapter 3, **A comparative study of some iteration processes**, we compare the rates of convergence of some iteration methods for contractions and show that the involved coefficients in such methods have an important role to play in determining the rate of convergence. By this study, we continue the research of Babu and Vara Prasad [5], Berinde [10, 11, 12, 13], Chugh and Kumar [17], Popescu [51], Thakur *et al.* [60].

The original contribution in this chapter is: Proposition 3.1, Proposition 3.2, Theorem 3.1, Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5, Example 3.1, Example 3.2, Example 3.3, Example 3.4.

They are published in [22] (Fathollahi, S, Ghiura, A, Postolache, M, Rezapour, S: *A comparative study on the convergence rate of some iteration methods involving contractive mappings.* Fixed Point Theory Appl. 2015:234 (2015)).

Let (X, d) be a metric space, $x_0 \in X$ and $T \colon X \to X$ a selfmap. The Picard iteration is defined by

$$x_{n+1} = Tx_n$$

for all $n \geq 0$. Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be sequences in $[0, 1]$. Then the Mann iteration method is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{3}$$

for all $n \geq 0$ (for more information, see Mann [38]). Also, the Ishikawa iteration method is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{cases} \tag{4}$$

for all $n \geq 0$ (for more information, see Ishikawa [28]). The Noor iteration method is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{5}$$

for all $n \geq 0$ (for more information, see Noor [47]). In 2007, Agarwal *et al.* defined their new iteration method by

$$\begin{cases} x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{6}$$

for all $n \geq 0$ (for more information, see Agarwal *et al.* [2]). In 2014, Abbas *et al.* defined their new iteration method by

$$\begin{cases} x_{n+1} = (1-\alpha_n)T y_n + \alpha_n T z_n, \\ y_n = (1-\beta_n)T x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{7}$$

for all $n \geq 0$ (for more information, see Abbas and Nazir [1]). In 2014, Thakur *et al.* defined their new iteration method by

$$\begin{cases} x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{8}$$

for all $n \geq 0$ (for more information, see Thakur *et al.* [60]). Also, the Picard S-iteration was defined by

$$\begin{cases} x_{n+1} = T y_n, \\ y_n = (1-\beta_n)T x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{9}$$

for all $n \geq 0$ (for more information, see Gorsoy and Karakaya [25], Ozturk [50]).

Let $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ be two fixed point iteration procedures that converge to the same fixed point p and $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \geq 0$. If the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ converge to a and b respectively and $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$, then we say that $\{u_n\}_{n\geq 0}$ converges faster than $\{v_n\}_{n\geq 0}$ to p (see Berinde [10] and Thakur *et al.* [60]).

We show that choosing a type of sequence $\{\alpha_n\}_{n\geq 0}$ in the Mann iteration has a notable role to play in the rate of convergence of the sequence $\{x_n\}_{n\geq 0}$.

Proposition 0.1. *Let C be a nonempty, closed and convex subset of a Banach space X, $x_1 \in C$, $T\colon C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. Consider the first case for Mann iteration. If the coefficients of Tx_n are greater than the coefficients of x_n, that is $1-\alpha_n < \alpha_n$ for all $n \geq 0$ or equivalently $\{\alpha_n\}_{n\geq 0}$ is a sequence in $(\frac{1}{2},1)$, then the Mann iteration converges faster than the Mann iteration which the coefficients of x_n are greater than the coefficients of Tx_n.*

We can consider four cases for writing the Ishikawa iteration method. In next result, we indicate each case by different enumeration. Similar to the last result, we want to compare the Ishikawa iteration method with itself in the four possible cases.

Proposition 0.2. *Let C be a nonempty, closed and convex subset of a Banach space X, $x_0 \in C$, $T\colon C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. Consider the following cases of the Ishikawa iteration method:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{10}$$

and

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n) T y_n, \\ y_n = \beta_n x_n + (1-\beta_n) T x_n, \end{cases} \tag{11}$$

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = \beta_n x_n + (1-\beta_n) T x_n, \end{cases} \tag{12}$$

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n) T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{13}$$

for all $n \geq 0$. If $1-\alpha_n < \alpha_n$ and $1-\beta_n < \beta_n$ for all $n \geq 0$, then the case (10) converges faster than the others (11), (12), (13). In fact, the Ishikawa iteration method is faster whenever the coefficients of Ty_n and Tx_n simultaneously are greater than the related coefficients of x_n for all $n \geq 0$.

Now consider eight cases for writing the Noor iteration method. By using a condition, we show that the coefficient sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ have effective roles to play in the rate of convergence of the sequence $\{x_n\}_{n\geq 0}$ in the Noor iteration method.

Theorem 0.10. *Let C be a nonempty, closed and convex subset of a Banach space X, $x_0 \in C$, $T\colon C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T.*

Consider the case (5) *of the Noor iteration method*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases}$$

for all $n \geq 0$. *If* $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$ *and* $1-\gamma_n < \gamma_n$ *for all* $n \geq 0$, *then the iteration* (5) *is faster than the other possible cases.*

As we know, the Agarwal iteration method could be written in the following four cases:

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{14}$$

$$\begin{cases} x_{n+1} = \alpha_n Tx_n + (1-\alpha_n) T y_n, \\ y_n = \beta_n x_n + (1-\beta_n) T x_n, \end{cases} \tag{15}$$

$$\begin{cases} x_{n+1} = \alpha_n Tx_n + (1-\alpha_n) T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{16}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = \beta_n x_n + (1-\beta_n) T x_n \end{cases} \tag{17}$$

for all $n \geq 0$. One can easily show that case (14) converges faster than the other ones for contractive maps. We record it as next Lemma.

Lemma 0.6. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $x_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$ *and* p *a fixed point of* T. *If* $1-\alpha_n < \alpha_n$ *and* $1-\beta_n < \beta_n$ *for all* $n \geq 0$, *then case* (14) *converges faster than* (15), (16) *and* (17).

Similar to Theorem 0.10, we can prove that for contractive maps one case in the Abbas iteration method converges faster than the other possible cases whenever the elements of the sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are in $(\frac{1}{2}, 1)$ for sufficiently large n.

Lemma 0.7. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $u_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$ *and* p *a fixed point of* T.

Consider the following case in the Abbas iteration method:

$$\begin{cases} u_{n+1} = \alpha_n T v_n + (1-\alpha_n) T w_n, \\ v_n = (1-\beta_n) T u_n + \beta_n T w_n, \\ w_n = (1-\gamma_n) u_n + \gamma_n T u_n \end{cases} \tag{18}$$

for all n. *If* $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$ *and* $1-\gamma_n < \gamma_n$ *for sufficiently large* n, *then case* (18) *converges faster than the other possible cases.*

Also, one can show that for contractive maps case (8) of the Thakur-Thakur-Postolache iteration method converges faster than the other possible cases whenever elements of the sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are in $(\frac{1}{2}, 1)$ for sufficiently large n. We record this result as follows.

Lemma 0.8. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $x_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$ *and* p *a fixed point of* T. *If* $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$ *and* $1-\gamma_n < \gamma_n$ *for sufficiently large* n, *then case* (8) *in the Thakur-Thakur-Postolache iteration method converges faster than the other possible cases.*

Finally, we have a similar situation for the Picard S-iteration which we record here.

Lemma 0.9. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $x_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$ *and* p *a fixed point of* T. *If* $1-\alpha_n < \alpha_n$ *and* $1-\beta_n < \beta_n$ *for sufficiently large* n, *then case* (9) *in the Picard S-iteration method converges faster than the other possible cases.*

In the next section, we compare the rate of convergence of some different iteration methods for contractive maps. Our goal is to show that the rate of convergence relates to the coefficients.

Theorem 0.11. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $u_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$ *and* p *a fixed point of* T. *Consider case* (7) *in the Abbas iteration method*

$$\begin{cases} u_{n+1} = (1-\alpha_n) T v_n + \alpha_n T w_n, \\ v_n = (1-\beta_n) T u_n + \beta_n T w_n, \\ w_n = (1-\gamma_n) u_n + \gamma_n T u_n, \end{cases}$$

case (18) *in the Abbas iteration method*

$$\begin{cases} u_{n+1} = \alpha_n T v_n + (1-\alpha_n) T w_n, \\ v_n = (1-\beta_n) T u_n + \beta_n T w_n, \\ w_n = (1-\gamma_n) u_n + \gamma_n T u_n, \end{cases}$$

and case (8) *in the Thakur-Thakur-Postolache iteration method*

$$\begin{cases} u_{n+1} = (1-\alpha_n) T u_n + \alpha_n T v_n, \\ v_n = (1-\beta_n) w_n + \beta_n T w_n, \\ w_n = (1-\gamma_n) u_n + \gamma_n T u_n \end{cases}$$

for all $n \geq 0$. *If* $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$ *and* $1-\gamma_n < \gamma_n$ *for sufficiently large* n, *then case* (18) *in the Abbas iteration method converges faster than case* (8) *in the Thakur-Thakur-Postolache iteration method. Also, case* (8) *in the Thakur-Thakur-Postolache iteration method is faster than case* (7) *in the Abbas iteration method.*

By using a similar proof, one can check the next result.

Theorem 0.12. *Let* C *be a nonempty, closed and convex subset of a Banach space* X, $x_1 \in C$, $T\colon C \to C$ *a contraction with constant* $k \in (0,1)$, p *a fixed point of* T *and* $\alpha_n, \beta_n, \gamma_n \in (0,1)$ *for all* $n \geq 0$. *Then case* (6) *in the Agarwal iteration method is faster than case* (3) *in the Mann iteration method, case* (7) *in the Abbas iteration method is faster than case* (3) *in the Mann iteration method, case* (8) *in the Thakur-Thakur-Postolache iteration method is faster than case* (3) *in the Mann iteration method, case* (6) *in the Agarwal iteration method is faster than case* (4) *in the Ishikawa iteration method, case* (7) *in the Abbas iteration method is faster than case* (4) *in the Ishikawa iteration method and case* (8) *in the Thakur-Thakur-Postolache iteration method is faster than case* (4) *in the Ishikawa iteration method.*

For examples and figures illustrating these results, please see [22].

In Chapter 4, **Iterative algorithms for a class of quasi variational inequalities** we introduce and study a new class of quasi variational inequalities, known as multivalued extended general quasi variational inequalities. It is shown that the multivalued extended general quasi variational inequalities are equivalent to the fixed point problems. We use this alternative equivalent formulation to suggest and analyze some iterative methods. We also introduce a new class of Wiener-Hopf equations, known as multivalued extended general implicit Wiener-Hopf equations. We establish the equivalence between the multivalued extended general quasi variational inequalities and multivalued extended general implicit Wiener-Hopf equations. Using this equivalence, we suggest and analyze some iterative methods. The results in this chapter follow the results of Stampacchia [57], Shi [56], Noor *et al.* [44, 45, 46].

Our original contribution in this chapter is: Theorem 4.1, Algorithm 4.1, Algorithm 4.2, Algorithm 4.3, Algorithm 4.4, Algorithm 4.5, Algorithm 4.6, Theorem 4.2, Corollary 4.1, Algorithm 4.7, Algorithm 4.8, Algorithm 4.9, Algorithm 4.10, Algorithm 4.11, Algorithm 4.12 and Theorem 4.3.

They are published in [48] (Noor, MA, Noor, KI, Khan, AG, Ghiura, A: *Iterative algorithms for solving a class of quasi variational inequalities.* U.P.B. Sci. Bull., Series A, 78(3), 3-18 (2016)).

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C(H)$ be a family of all nonempty compact subsets of H. Let $T,\ V\colon H\to C(H)$ be the multivalued operators. Let $h_1,\ h_2\colon H\to H$ and $N(\cdot,\cdot):H\times H\to H$ be the single valued operators.

Given a point-to-set mapping $\Omega\colon u\to\Omega(u)$, which associates a closed convex valued set $\Omega(u)$ with any element $u\in H$, we consider problem of finding $u,\ w,\ y\in H : w\in T(u),\ y\in V(u),\ h_1(u),\ h_2(u)\in\Omega(u)$, and

$$\langle \rho N(w,y)+h_2(u)-h_1(u),h_1(v)-h_2(u)\rangle\geq 0,\quad \forall v\in H : h_1(v)\in\Omega(u) \tag{19}$$

where $\rho>0$, is a constant. Problem (19) is called the multivalued extended general quasi-variational inequality. It has many applications in the field of mechanics, physics, pure and applied sciences, see: Facchinei *et al.* [20], Giannessi and Maugeri [23], Kravchuk and Neittaanmaki [34], Lenzen *et al.* [35], Liu and Cao [36] and references therein.

Lemma 0.10. *For a given $z\in H,\ u\in\Omega$ satisfies the inequality*

$$\langle u-z,v-u\rangle\geq 0,\quad \forall v\in\Omega,$$

if and only if

$$u=P_\Omega[z],$$

where P_Ω is the projection of H into a closed and convex set Ω.

We now define the concept of strong monotonicity for the bifunction operator $N(\cdot,\cdot)$, which was introduced by Noor [42].

Definition 0.7. The single valued operator $N(\cdot,\cdot)$ is said to be strongly monotone with respect to the first argument if, for all $u_1,\ u_2\in H$, there exists a constant $\alpha>0$, such that

$$\langle N(w_1,\cdot)-N(w_2,\cdot),u_1-u_2\rangle\geq\alpha\|u_1-u_2\|^2,\quad \forall w_1\in T(u_1),\ w_2\in T(u_2).$$

Definition 0.8. The single valued operator $N(\cdot,\cdot)$ is said to be Lipschitz continuous with respect to the first argument, if there exists a constant $\beta > 0$, such that

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leq \beta\|u_1 - u_2\|, \quad \forall u_1,\ u_2 \in H.$$

Similarly, we can define the strong monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument.

Definition 0.9. The set valued operator $V\colon H \to C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$M(V(u_1), V(u_2)) \leq \xi\|u_1 - u_2\|, \quad \forall u_1,\ u_2 \in H,$$

where $C(H)$ is the family of all nonempty compact subsets of H and $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$, that is for any two nonempty subsets A and B of H,

$$M(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\},$$

where

$$d(x,B) = \inf_{y\in B}\|x-y\| \quad \text{and} \quad d(A,y) = \inf_{x\in A}\|x-y\|.$$

In order to prove our main results, the next lemma is very important.

Lemma 0.11 ([40]). *Let (H,d) be a complete metric space, $T\colon H \to CB(H)$ be a set-valued mapping. Then, for all x, $y \in H$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\|u - v\| \leq M(T(x), T(y)).$$

In this section, we show that the multivalued extended general quasi-variational inequality (19) is equivalent to a fixed point problem using Lemma 0.10. We use this alternative equivalent formulation to discuss the existence of a solution to problem (19).

Lemma 0.12. *Let $\Omega(u)$ be a closed and convex valued set in H. Then $u, w, y \in H$ is a solution to (19) if and only if $u, w, y \in H$ satisfies the relation*

$$h_2(u) = P_{\Omega(u)}[h_1(u) - \rho N(w,y)],$$

where $\rho > 0$ is a constant and $P_{\Omega(u)}$ is the projection of H onto the closed convex-valued set $\Omega(u)$.

Assumption 0.1. *For a constant $\nu > 0$, the implicit projection operator $P_{\Omega(u)}$ satisfies the condition*

$$\|P_{\Omega(u)}[w] - P_{\Omega(v)}[w]\| \leq \nu\|u - v\|, \quad \textit{for all } u, v, w \in H.$$

We now discuss the existence of a solution to problem (19) and this is the main motivation of our next result.

Theorem 0.13. *Let $\Omega(u)$ be a closed and convex valued set in H. Let the operator $N(\cdot,\cdot)$ be strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with respect to the first argument with constant $\beta > 0$. Let operators h_1, $h_2\colon H \to H$ be strongly monotone with constants $\sigma_1 > 0$, $\sigma_2 > 0$ and Lipschitz continuous with constants $\delta_1 > 0$, $\delta_2 > 0$, respectively. Assume that the operator $N(\cdot,\cdot)$ is Lipschitz continuous with respect to the second argument with constant $\eta > 0$. Let T, $V\colon H \to C(H)$ are M-Lipschitz continuous mappings with constants $\mu > 0$ and $\xi > 0$ respectively. If Assumption 0.1 holds and*

$$\theta = k + t(\rho) + \rho\eta\xi < 1, \tag{20}$$

and

$$\begin{aligned} k &= \nu + \sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\sigma_2 + \delta_2^2}, \\ t(\rho) &= \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}, \end{aligned}$$

then there exists a solution u, w, $y \in H : w \in T(u)$, $y \in V(u)$, and $h_1(u), h_2(u) \in \Omega(u)$ satisfying the problem (19).

In the next section, we develop and discuss some iterative methods for solving problem (19). We also consider the convergence analysis of these iterative methods.

Algorithm 0.1. *Assume T, $V\colon H \to C(H)$ be multivalued operators. Suppose that $N\colon H \times H \to H$, h_1, $h_2\colon H \to H$ are single valued operators. Let $\Omega(u)$ be a closed convex valued set in a real Hilbert space H. For given u_0, w_0, $y_0 \in H$, let $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $h_1(u_0) \in \Omega(u_0)$, $h_2(u_0) \in \Omega(u_0)$ and*

$$u_1 = (1-\lambda) u_0 + \lambda \left\{ u_0 - h_2(u_0) + P_{\Omega(u_0)} \left[h_1(u_0) - \rho N(w_0, y_0) \right] \right\}.$$

Using Lemma 0.11; since $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, then there exist $w_1 \in T(u_1)$, $y_1 \in V(u_1)$ such that

$$\|w_0 - w_1\| \le M(T(u_0), T(u_1))$$

$$\|y_0 - y_1\| \le M(V(u_0), V(u_1)),$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$. Let

$$u_2 = (1-\lambda) u_1 + \lambda \left\{ u_1 - h_2(u_1) + P_{\Omega(u_1)} \left[h_1(u_1) - \rho N(w_1, y_1) \right] \right\}.$$

By continuing this process, we can obtain the sequences $\{u_n\}, \{w_n\}, \{y_n\}$ *such that*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$

$$u_{n+1} = (1-\lambda) u_n + \lambda \left\{u_n - h_2(u_n) + P_{\Omega(u_n)}[h_1(u_n) - \rho N(w_n, y_n)]\right\},$$

for $n = 0, 1, 2, \ldots$.

In the next theorem, we show that the approximate solution obtained from the iterative Algorithm 0.1 converges strongly to $u, w, y \in H$, the exact solution to problem (19).

Theorem 0.14. *Let* $\Omega(u)$ *be any closed and convex valued set in* H*. Let the operator* $N(\cdot,\cdot)$ *be strongly monotone with respect to the first argument with constant* $\alpha > 0$ *and Lipschitz continuous with respect to the first argument with constant* $\beta > 0$*. Let the operators* h_1, $h_2 \colon H \to H$ *be strongly monotone with constant* $\sigma_1 > 0$, $\sigma_2 > 0$ *and Lipschitz continuous with constants* $\delta_1 > 0$, $\delta_2 > 0$*, respectively. Assume that the operator* $N(\cdot,\cdot)$ *is Lipschitz continuous with respect to the second argument with constant* $\eta > 0$*. Let* T*,* $V \colon H \to C(H)$ *be M-Lispchitz continuous mappings with constants* $\mu > 0$ *and* $\xi > 0$ *respectively. If Assumption* 0.1 *and relation* (20) *hold, then there exists a solution* u*,* w*,* $y \in H : w \in T(u)$*,* $y \in V(u)$*, and* $h_1(u)$*,* $h_2(u) \in \Omega(u)$ *satisfying problem* (19)*, and the sequences* $\{u_n\}$*,* $\{w_n\}$*, and* $\{y_n\}$ *generated by Algorithm* 0.1 *converges to* u*,* w *and* y *strongly in* H*, respectively.*

Now, we introduce a new class of Wiener-Hopf equations, which is called the multivalued extended general implicit Wiener-Hopf equations. We establish the equivalence between the multivalued extended general implicit Wiener-Hopf equations and problem (19). By using this equivalence, we suggest a number of new iterative methods for solving the different classes of problem (19) and its variant forms.

For given nonlinear multivalued operators T, $V \colon H \to C(H)$ and single valued operators $N(\cdot,\cdot) : H \times H \to H$, and h_1, $h_2 \colon H \to H$. Suppose that the inverse of the operator h_2 exists, we consider the problem of finding z, u, w, $y \in H : w \in T(u)$, $y \in V(u)$, and

$$N(w, y) + \rho^{-1} Q_{\Omega(u)}[z] = 0, \tag{21}$$

where $Q_{\Omega(u)} = I - h_1\left(h_2^{-1} P_{\Omega(u)}\right)$, I is the identity operator and $\rho > 0$ is a constant. The equation (21) is known as multivalued extended general implicit Wiener-Hopf equations.

Lemma 0.13. *The problem* (19) *has a solution* $u,\ w,\ y \in H : w \in T(u),\ y \in V(u),$ *and* $h_1(u),\ h_2(u) \in \Omega(u)$*, if and only if problem* (21) *has a solution* $z,\ u,\ w,\ y \in H :$ $w \in T(u),\ y \in V(u)$*, provided*

$$h_2(u) = P_{\Omega(u)}[z],$$

and

$$z = h_1(u) - \rho N(w, y),$$

where $\rho > 0$ *is a constant.*

Lemma 0.13 implies that problem (19) and problem (21) are equivalent. This equivalent formulation is used to suggest and analyze some iterative methods for solving (19).

Algorithm 0.2. *For given* $z_0,\ u_0,\ w_0,\ y_0 \in H : w_0 \in T(u_0),\ y_0 \in V(u_0)$*, compute the sequences* $\{z_n\},\ \{u_n\},\ \{w_n\}$*, and* $\{y_n\}$ *by the iterative schemes*

$$h_2(u_n) = P_{\Omega(u_n)}[z_n]$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$

$$z_{n+1} = h_1(u_n) - \rho N(w_n, y_n), \quad n = 0, 1, 2, \ldots.$$

We now discuss the convergence analysis of Algorithm 0.2 and this is the main motivation of our next result.

Theorem 0.15. *With the same conditions as in Theorem* 0.13*, there exist* $z,\ u,\ w,$ $y \in H : w \in T(u)$*, and* $y \in V(u)$ *satisfying problem* (21) *and the sequences* $\{z_n\}$, $\{u_n\}$, $\{w_n\}$*, and* $\{y_n\}$ *generated by Algorithm* 0.2 *converge to* z*,* u*,* w*, and* y *strongly in* H*, respectively.*

For examples, corollaries and special cases of these algorithms, which are used to solve some important classes of quasi variational inequalities, please see [48].

References

[1] Abbas, M, Nazir, T: A new faster iteration process applied to constrained minimization and feasibility problems. Mat. Vesn. **66**(2), 223-234 (2014)

[2] Agarwal, RP, O'Regan, D, Sahu, DR: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. **8**(1), 61-79 (2007)

[3] Akbulut, S, Ozdemir, M: Picard iteration converges faster than Noor iteration for a class of quasi-contractive operators. Chiang Mai J. Sci. **39**(4), 688-692 (2012)

[4] Argyros, IK: Iterations converging faster than Newton's method to the solutions of nonlinear equations in Banach space. Ann. Univ. Sci. Bp. Rolando Eötvös Nomin., Sect. Comput. **11**, 97-104 (1991)

[5] Babu, GVR, Vara Prasad, KNVV: Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators. Fixed Point Theory Appl. **2006**, Article ID 49615 (2006)

[6] Baiocchi, A, Capelo, A: Variational and Quasi-Variational Inequalities. J Wiley and Sons. New York (1984)

[7] Bakhtin, IA: The contraction mapping principle in quasimetric spaces. Funct. Anal. **30**, 26-37 (1989)

[8] Banach, S: Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrals. Fundam. Math. **3**, 133-181 (1922)

[9] Batul, S, Kamran, T: C^*-Valued contractive type mappings. Fixed Point Theory Appl. 2015:142 (2015)

[10] Berinde, V: Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Fixed Point Theory Appl. **2004**(2), 97-105 (2004)

[11] Berinde, V, Berinde, M: The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings. Carpath. J. Math. **21**(1-2), 13-20 (2005)

[12] Berinde, V: A convergence theorem for Mann iteration in the class of Zamfirescu operators. An. Univ. Vest Timis., Ser. Mat.-Inform. **45**(1), 33-41 (2007)

[13] Berinde, V: Iterative Approximation of Fixed Points. Springer, Berlin (2007)

[14] Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169-179 (1993)

[15] Chidume, CE, Ofoedu, EU, Zegeye, H: Strong and weak convergence theorems for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **280**, 364-374 (2003)

[16] Chidume, CE, Shahzad, N, Zegeye H: Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense. Numer. Funct. Anal. Optim. **25**(3-4), 239-257 (2004)

[17] Chugh, R, Kumar, S: On the rate of convergence of some new modified iterative schemes. Am. J. Comput. Math. **3**, 270-290 (2013)

[18] Czerwick, S: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)

[19] Czerwick, S: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**, 263-276 (1998)

[20] Facchinei, F, Kanzow, C, Sagratella, S: Solving quasi-variational inequalities via their KKT conditions. Math. Program Ser. A. **144**(1),369-412 (2014)

[21] Falset, JG, Kaczor, W, Kuczumow, T, Reich, S: Weak convergence theorems for asymptotically nonexpansive mappings and semigroups. Nonlinear Anal. **43**(3), 377-401 (2001)

[22] Fathollahi, S, **Ghiura, A**, Postolache, M, Rezapour, S: A comparative study on the convergence rate of some iteration methods involving contractive mappings. Fixed Point Theory Appl. 2015:234 (2015)

[23] Giannessi, F, Maugeri, A: Variational Inequalities and Network Equilibrium Problems. Plenum Press, New York (1995)

[24] Glowinski, R, Lions, JL, Tremolieres, R: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)

[25] Gorsoy, F, Karakaya, V: A Picard S-hybrid type iteration method for solving a differential equation with retarded argument (2014). arXiv: 1403.2546v2 [math.FA]

[26] Guo, WP, Guo, W: Weak convergence theorems for asymptotically nonexpansive non-self mappings, Appl. Math. Lett. **24**, 2181-2185 (2011)

[27] Guo, WP, Cho, YJ, Guo, W: Convergence theorems for mixed type asymptotically nonexpansive mappings, Fixed Point Theory Appl. 2012:224 (2012)

[28] Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)

[29] Kamran, T, Postolache, M, **Ghiura, A**, Batul, S, Ali, R: The Banach contraction principle in C^*-algebra-valued b-metric spaces with application. Fixed Point Theory Appl. 2016:10 (2016)

[30] Khamsi, MA, Kirk, W: An Introduction to Metric Spaces and Fixed Point Theory. A Wiley - Interscience Publication (2001)

[31] Khamsi, MA: Remarks on Carisit's fixed point theorem. Nonlinear Anal. **71**(1-2), 227-231 (2009)

[32] Kinderlehrer, D, Stampacchia, G: An Introduction to Variational Inequalities and their Applications. Academic Press, London, England (1981)

[33] Kirk, W, Shahzad, N: Fixed Point Theory in Distance Spaces. Springer, Berlin (2014)

[34] Kravchuk, AS, Neittaanmaki, PJ: Variational and Quasi Variational Inequalities in Mechanics. Springer, Dordrecht, Holland (2007)

[35] Lenzen, F, Becker, F, Lellmann, J, Petra, S, Schnorr, CA: Class of quasi-variational inequalities for adaptive image denoising and decomposition. Comput Optim. Appl. **54**, 371-398 (2013)

[36] Liu, Q, Cao, A: A recurrent neural network based on projection operator for extended general variational inequalities. IEEE Trans. Syst. Man. Cybers. Part B Cyber **40**, 928-938 (2010)

[37] Ma, Z, Jiang, L, Sun, H: C^*-algebra-valued metric spaces and related fixed point theorems. Fixed Point Theory Appl. **2014**, 206 (2014)

[38] Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)

[39] Murphy, GJ: C^*-Algebras and Operator Theory. Academic Press, London (1990)

[40] Nadler Jr, SB: Multi-valued contraction mappings. Pacific J. Math. **30**, 475-488 (1969)

[41] Noor, MA: Wiener-Hopf equations and variational inequalities. J. Optim Theory Appl. **79**, 197-206 (1993)

[42] Noor, MA: Generalized multivalued quasi-variational inequalities (II). Comput Math. Appl. **35**, 63-78 (1998)

[43] Noor, MA: Variational Inequalities and Applications. Lectures Notes, Mathematics Department, COMSTAS Institute of Information Technology,Islamabad, Pakistan (2009-2013)

[44] Noor, MA, Noor, KI: Sensitivity analysis of some quasi variational inequalities. J. Adv. Math. Stud. **6**, 43-52 (2013)

[45] Noor, MA, Noor, KI, Khan, AG: Some iterative schemes for solving extended general quasi variational inequalities. Appl. Math. Inf. Sci. **7**, 917-925 (2013)

[46] Noor, MA, Noor, KI, Khan, AG: Dynamical systems for quasi variational inequalities. Ann. Funct. Anal. **6**, 193-209 (2015)

[47] Noor, MA: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**, 217-229 (2000)

[48] Noor, MA, Noor, KI, Khan, AG, **Ghiura, A**: Iterative algorithms for solving a class of quasi variational inequalities. U.P.B. Sci. Bull. Ser. A **78**(3), 3-18 (2016)

[49] Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Amer. Math. Soc. **73**, 591-597 (1967)

[50] Ozturk Celikler, F: Convergence analysis for a modified SP iterative method. Sci. World J. **2014**, Article ID 840504 (2014)

[51] Popescu, O: Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Math. Commun. **12**(2), 195-202 (2007)

[52] Saluja, GS: Convergence theorems for two asymptotically nonexpansive non-self mappings in uniformly convex Banach spaces. J. Indian Math. Soc. **81**(3-4), 369-385 (2014)

[53] Saluja, GS, Postolache, M, **Ghiura, A**: Convergence theorems for mixed type asymptotically nonexpansive mappings in the intermediate sense. J. Nonlinear Sci. Appl. **9**, 5119-5135 (2016)

[54] Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Austral. Math. Soc. **43**(1), 153-159 (1991)

[55] Shehwar, D, Batul, S, Kamran, T, **Ghiura, A**: Caristi's fixed point theorem on C^*-algebra-valued metric spaces. J. Nonlinear Sci. Appl. **9**, 584-588 (2016)

[56] Shi, P: Equivalence of variational inequalities with Wiener-Hopf equations. Proc. Amer. Math. Soc. **111**, 339-346 (1991)

[57] Stampacchia, G: Formes bilineaires coercitives sur les ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413-4416 (1964)

[58] Takahashi, W, Kim, GE: Approximating fixed points of nonexpansive mappings in Banach spaces. Math. Japonoica **48**(1), 1-9 (1998)

[59] Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)

[60] Thakur, D, Thakur, BS, Postolache, M: New iteration scheme for numerical reckoning fixed points of nonexpansive mappings. J. Inequal. Appl. 2014:328 (2014)

[61] Wang, L: Strong and weak convergence theorems for common fixed point of non-self asymptotically nonexpansive mappings. J. Math. Anal. Appl. **323**(1), 550-557 (2006)

[62] Wei, S, Guo, WP: Strong convergence theorems for mixed type asymptotically nonexpansive mappings. Comm. Math. Res. **31**, 149-160 (2015)

[63] Wei, S, Guo, WP: Weak convergence theorems for mixed type asymptotically nonexpansive mappings. J. Math. Study **48**(3), 256-264 (2015)