University Politehnica of Bucharest
Faculty Of **Applied Sciences**
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Summary of PhD Thesis

Optimization Techniques and Methods in Reliability Allocation

Metode şi Tehnici de Optimizare în Alocarea Fiabilităţii

(Rezumat)

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Thesis Abstract

This Thesis focuses on very actual problems in optimization techniques, namely methods in reliability allocation. We look optimal reliability allocation problem as a mathematical problem though its origins are in engineering. The assignation of reliability values between the various subsystems and elements can be made on the basis of complexity, criticality, estimated achievable reliability, or any other factors considered appropriate by the analyst making the allocation.

The first steps for this purpose involve the study of multivariate reliability polynomial. The algebraic-geometric properties of this polynomial include: homogenization of a reliability polynomial, compact hypersurfaces attached to homogeneous polynomials, an affine diffeomorphism that preserves a reliability polynomial, duality of networks via a diffeomorphism, straight line segments in constant level sets associated to a reliability polynomial. The problem of affine diffeomorphisms that preserves a reliability polynomial is yet an open geometrical problem.

A system (mechanical, electrical, computer hardware and software etc) is generally designed as an assembly of subsystems, each with its own reliability attributes. The cost of the system is the sum of the costs for all the subsystems. Reliability allocations are used to set the goals for various subsystem or functional blocks such that the overall system level reliability can be achieved in an effective way. Our model discussed also the posynomial cost function, taking into account all its properties regarding multivariable monotony and convexity (either Euclidean or with respect to a connection). Such a cost and the reliability constraint, associated to reliability polynomial, lead us to geometric programming method. Especially it is examined the optimal reliability approaches to allocate the reliability values based on minimization of the total cost for a series-parallel systems. The problem is approached as a nonlinear programming problem and general costs formulas are suggested. In this approach, there are highlighted dualities between reliability systems and electric circuits.

Another basic subject is an optimal control problem whose Bolza payoff is the sum between a simple integral and a function of the initial and final events and whose evolution ODE is a reliability flow. The obtained results include a list of payoffs with reliability sense, optimal value of mean time to failure functional constrained by controlled queueing system, optimal value of mean time to failure functional constrained

by controlled Markov system, and optimal value of mean time to failure functional associated to parallel systems.

The technological evolution of today requires development of the methods in reliability allocation, and our study highlights everywhere possible developments.

REZUMATUL TEZEI

Această Teză se concentrează asupra unor probleme foarte actuale în tehnicile de optimizare, şi anume metodele de alocare a fiabilităţii. Privim problema de alocare optimă a fiabilităţii ca o problemă matematică, deşi originea sa este în inginerie. Alocarea valorilor de fiabilitate între diferite subsisteme şi elemente poate fi făcută pe baza unor factori consideraţi adecvaţi de către analistul care face alocarea.

Primii paşi în acest scop implică studierea polinoamelor de fiabilitate multivariabilă. Proprietăţile algebrico-geometrice ale acestor polinoame includ: omogenizarea unui polinom de fiabilitate, hipersuprafeţe compacte ataşate polinoamelor omogene, un difeomorfism afin care păstrează un polinom de fiabilitate, dualitatea reţelelor printr-un difeomorfism, segmente de dreaptă aflate în mulţimile de nivel constant asociate unui polinom de fiabilitate. Problema difeomorfismelor afine care păstrează un polinom de fiabilitate este încă o problemă geometrică deschisă.

Un sistem (mecanic, electric, hardware şi software pentru calculator etc.) este conceput în general ca un ansamblu de subsisteme, fiecare având propriile atribute de fiabilitate. Costul sistemului este suma costurilor subsistemelor. Alocările de fiabilitate sunt utilizate pentru a stabili obiectivele pentru diverse subsisteme sau blocuri funcţionale astfel încât nivelul general de fiabilitate al sistemului să poată fi realizat într-un mod eficient. Modelul nostru discută funcţia cost pozinomială, luând în considerare proprietăţile sale, fie euclidiene, fie cu privire la o conexiune. Un astfel de cost şi restricţile asociate polinomului de fiabilitate, ne conduc la metoda de programare geometrică. În special se examinează abordările optime pentru a aloca valorile de fiabilitate în vederea minimizării costului total pentru un sistem serie - paralel. Problema este abordată ca problemă de programare neliniară şi sunt sugerate diverse formule generale de costuri. În această abordare, sunt evidenţiate dualităţil dintre sistemele de fiabilitate şi circuitele electrice.

Un alt subiect este o problemă de control optimal de tip Bolza ale cărei ecuaţii de evoluţie este un "curent de fiabilitate". Rezultatele obţinute includ o listă de funcţii obiectiv cu semnificaţie în teoria fiabilităţii: funtionala timp mediu optim până la defecţiune constrânsâ de un sistem de aşteptare controlat, funtionala timp mediu optim până la defecţiune constrânsâ de un sistem Markov controlat şi funtionala

timp mediu optim până la defecţiune asociat sistemelor paralele.

Evoluţia tehnologică actuală necesită dezvoltarea metodelor de alocare a fiabilităţii, iar studiul nostru evidenţiază evoluţii posibile.

Content of the Thesis

1 Organization of the Thesis and original contributions

In this Thesis, some reliability optimization problems have been formulated and solved.

The entire Thesis has been divided into six (6) Chapters as follows:

Chapter 1: Introduction

Chapter 2: Problems on multivariate reliability polynomial

Chapter 3: Optimal reliability allocation

Chapter 4: Geometric programming approaches of reliability allocation

Chapter 5: Optimal reliability allocation for redundancy series - parallel systems

Chapter 6: Optimal control on reliability problems

Chapter 1, titled **Introduction**, summarizes references used throughout the work. Primarily the basic concepts, destinations of reliability systems with different types with their applications and also the different methods to maximize the reliability of a system are detailed in the first Chapter. Also the survey is carried out on reliability optimization problems by classifying the models under system problems. The survey of literature authenticates that very few research papers advocated the fact that "reliability optimization allocation, for simple and complex systems, with constraints and sensitivity analysis" can be studied and this has promoted the author to do research in this direction. This Chapter ends with a survey on the structure of the Thesis.

Chapter 2, titled **Problems on multivariate reliability polynomial**, focuses on an algebraic-geometric properties of the multivariate reliability polynomial. *Homogenization of multivariate reliability polynomial* refers to the fact that this process consists in to do homogenizing on each variable separately. The problem of *affine diffeomorphisms* that preserves a reliability polynomial is yet an open geometrical problem. Our model is a symmetry with respect to the origin and translation. The *dual networks via diffeomorphisms* remaind the idea of geometrical global equivalence. The problem of *straight line segments in constant level sets associated to a reliability polynomial* comes from a private discussion with Prof. Dr. Gheorghiţă Zbăganu. The original results in this Chapter are: Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4.

A property of homogeneous polynomials is stated in

Theorem 2.1 *If $f(x_1, ..., x_n)$ is a homogeneous polynomial of degree $p \geq 1$ which has at least a strictly positive value, then the set $M : f(x_1, ..., x_n) = 1$, is a hypersurface. If $f(x_1, ..., x_n) > 0$ excepting $(0, ..., 0)$, then the hypersurface M is compact and conversely.*

The following two theorems summarize our model of affine diffeomorphism that preserves a reliability polynomial.

Theorem 2.2 *The diffeomorphism $p^{e'} = 1 - p^e$ has two properties: (i) it transforms the unit hypercube $[0, 1]^n$ into itself (closed invariant set); (ii) it preserves the multivariate reliability polynomial if the graphs (V, A) and $(V, E \setminus A)$ are simultaneously connected.*

Theorem 2.3 *A series system G is dual to a parallel system G' via the diffeomorphism $R_{G'} = 1 - R_G$, $p^{e'} = 1 - p^e$, only if they have the same number of components.*

Theorem 2.4 *For $c > 0$, there exist no straight line segment in the constant level set $c = \prod_{e=1}^{n} p^e$.*

The original results will be published in [141] Udrişte, C., Ţevy, I., & Abed, S. A. (2017). *Problems on multivariate reliability polynomial.* Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat., accepted for publication.

Chapter 3, titled **Optimal reliability allocation**, a system (mechanical, electrical, computer hardware and software etc) is generally designed as an assembly of subsystems, each with its own reliability attributes. The cost of the system is the sum of the costs for all the subsystems. This Chapter examines possible approaches to allocate the reliability values based on minimization of the total cost on the intersection between Tzitzeica semispace and a unit hypercube. The original results include: (i) a critical point is a fixed point of a suitable application, (ii) theorems for restoring Riemannian convex functions; (iii) the first cost with exponential behavior is Euclidean convex; the second cost with exponential behavior is Riemannian convex; our particular posynomial cost is Euclidean convex, (iv) an additively decomposable cost function is convex on a product Riemannian manifold.

The original results in this Chapter are: Proposition 3.1, Proposition 3.2 and three significant reliability cost models.

Proposition 3.1. *If $(R_1, ..., R_n)$ is a solution of the nonlinear programming problem for the minimization of the total cost of a series system, then it is a fixed point of the application*

$$\left(\frac{\lambda \, R_G}{a_1 \, \frac{dC_1}{dR_1}}, ..., \frac{\lambda \, R_G}{a_n \, \frac{dC_n}{dR_n}} \right).$$

Proposition 3.2 *The total cost* $C(R_1, ..., R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ *is convex on the Riemannian manifold* $(\mathbb{R} \times ... \times \mathbb{R}, \oplus_{i=1}^{n} g_i(R_i))$.

Three significant reliability cost models

There is always a cost associated with changing a design, use of high quality materials, retooling costs, administrative fees, or other factors. The cost increases as the allocated reliability approaches the maximum achievable reliability. This is a reliability value that is approached asymptotically as the cost increases but is never actually reached. The cost increases as the allocated reliability departs from the minimum or current value of reliability. It is assumed that the reliabilities for the components will not take values any lower than they already have. Depending on the optimization, a component's reliability may not need to be increased from its current value but it will not drop any lower. The cost is a function of the range of improvement, which is the difference between the component's initial reliability and the corresponding maximum achievable reliability. Before attempting at improving the reliability, the cost as a function of reliability for each component must be obtained. Otherwise, the design changes may result in a system that is needlessly expensive or over-designed. Development of the cost of reliability relationship offers to mathematicians an understanding of which components or subsystems must be improve. The first step is to obtain a relationship between the cost of improvement and reliability. The second step is to model the cost as a function of reliability. The preferred approach would be to formulate the cost function from actual cost data. This can be done taking the past data. However, there are many cases where no such information is available. For this reason, a general behavior model of the cost versus the component reliability can be developed for performing reliability optimization. The objective of cost functions is to model an overall cost behavior for all types of components. But, it is impossible to formulate a model that is precisely applicable to every situation. However, one of the reliability cost models available can be used depending on situation. All these models can be tried and one which is suitable to component or situation can be adopted.

1. Exponential behavior model

Let $0 < R_i < 1,\ i = 1, 2, ..., n$, and a_i, b_i be constants. The most important cost function has an exponential behavior. It was proposed by [58] (see also [84]) in the form

$$C_i(R_i) = a_i\ exp\left(\frac{b_i}{1 - R_i}\right),\ i = 1, 2, ..., n.$$

Let $a_i > 0$, $b_i > 0$.

Then, each $C_i(R_i)$ is an increasing and convex function (in Euclidean sense), and $\lim_{R_i \to 1} C_i(R_i) = \infty$.

The total cost $C(R_1, ..., R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ has similar properties.

2. Exponential behavior model with feasibility factor

Let $0 < f_i < 1$ be a *feasibility factor*, $R_{i,min}$ be *minimum reliability* and $R_{i,max}$ be *maximum reliability*. Another important cost function, with exponential behavior, is given by

$$C_i(R_i) = exp\left((1 - f_i)\frac{R_i - R_{i,min}}{R_{i,max} - R_i}\right), \; R_{i,min} \le R_i \le R_{i,max}, \; i = 1, 2, ..., n.$$

In this case:

(1) the graph of $C_i(R_i)$ has an inflection point at $R_i = \frac{(1+f_i)R_{i,max}+(1-f_i)R_{i,min}}{2}$;

(2) each $C_i(R_i)$ is a convex function (in Euclidean sense) for $R_i > \frac{(1+f_i)R_{i,max}+(1-f_i)R_{i,min}}{2}$;

(3) each $C_i(R_i)$ is a concave function (in Euclidean sense) for $R_i < \frac{(1+f_i)R_{i,max}+(1-f_i)R_{i,min}}{2}$.

The total cost $(R_1, ..., R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ has similar properties.

Let us find a Riemannian metric $g_i(R_i)$ on $\mathbb{R}$ such that the function $C_i(R_i)$ to be convex on the Riemannian manifold $(\mathbb{R}, g_i(R_i))$. According the previous section, and the Euclidean Hessian, it is enough to fix the connection $\Gamma_i(R_i) = \frac{2(1-f_i)}{R_{i,max}-R_i}$ and hence $g_i(R_i) = (R_{i,mx} - R_i)^{-4(1-f_i)}$.

3. Posynomial behavior model

An example that differs fundamentally from the previous ones is given by the posynomial cost (see also [140])

$$C(R) = c_1R_1 + c_2R_2 + c_3R_1^{-a}R_2^{-b}, \; 0 < R_i \le 1, \; c_i > 0, \; a > 0, b > 0.$$

The minimum point of $C(R)$ satisfies also the system

$$c_1R_1 = \frac{a}{a+b+1}M, \; c_2R_2 = \frac{b}{a+b+1}M, \; c_3R_1^{-a}R_2^{-b} = \frac{1}{a+b+1}M.$$

Part of this Chapter was presented at The International Conference on Applied Mathematics and Numerical Methods (ICAMNM), University of Craiova, April 14-16, (2016), Craiova, Romania. The original results are published in [136] Udrişte, C., Abed, S. A., & Rasheed, A. S. (2016). *Optimal reliability allocation.* American Review of Mathematics and Statistics, DOI: 10.15640/arms.v4n2a9, 4(2), 82-91.

Chapter 4, titled **Geometric programming approaches of reliability allocation**, every system has a reliability goal that needs to be achieved. Our model discussed the posynomial cost function, taking into account all its properties regarding multivariate monotony and convexity (either Euclidean or with respect to a connection). Such a cost and the reliability constraint, associated to reliability polynomial, lead us to geometric programming method. Some examples that illustrate the results are given.

The original results in this Chapter are: Theorem 4.1, Theorem 4.2, Remark 4.1, Theorem 4.3 and geometric programming models.

New properties of monomials are given by

Theorem 4.1 *The general monomial* $\psi(x) = x_1^{a_1} \cdots x_m^{a_m}$ *is: (i) monotonically increasing in each variable, if* $a_1 > 0, \dots a_m > 0$; *(ii) multi-monotonically increasing, i.e.,*

$$(x_1, \dots, x_m) \le (y_1, \dots y_m) \Rightarrow \sum_{A \subset \mathbb{N}_m} (-1)^{card\, A} \prod_{i \in A} x_i^{a_i} \prod_{j \in \mathbb{N}_m \setminus A} y_j^{a_j} \ge 0$$

iff $a_1 \cdots a_m > 0$*; (iii) monotonically of Lebesgue type, i.e., for each domain* $\mathbb{D} \in \mathbb{R}_+^2$, *the function attains the extremum values on boundary* $\partial\mathbb{D}$.

Theorem 4.2 *There exist an infinity of linear symmetric connections* $\Gamma^i_{jk}(x)$ *on* $\mathbb{R}_+^m$ *such that the monomial* $\psi(x)$ *to be convex.*

The duality between series and parallel system stated in Theorem 2.3, gives us

Theorem 4.3 *The geometric program for a parallel system is equivalent to the geometric program for a series system.*

An original idea is highlighted in **Remark 4.1** whose final words are: *we should mention that does not matter how we obtain the convexity since once it is created we have all ingredients for convex programming theory (see [134]).*

Part of this Chapter was presented at The X-th International Conference Differential Geometry and Dynamical System(DGDS-2016), Mangalia, August 28- 3 September, (2016), Mangalia, Romania. The original results are published in [137] Udrişte, C., Abed, S. A., & Ţevy, I. (2017). *Geometric programming approaches of reliability allocation.* U.P.B. Sci. Bull., Series A, 79(3), 3-10 (IF 0.365).

Chapter 5, titled **Optimal reliability allocation for redundancy series - parallel systems**, examines the optimal reliability approaches to allocate the reliability values based on minimization of the total cost for a series-parallel systems. The problem is approached as a nonlinear programming problem and general costs formulas were suggested. The original results include: (i) submersion of a "series-parallel system" into a "series system", (ii) detailed analyse of a series-parallel system whose components of each subsystem have the same reliability; (iii) designing series-parallel systems by similarities with other engineering problems; (iv) dualities between reliability systems and electric circuits. Examples to illustrate the results are given.

The original results in this Chapter are: Theorem 5.1, Corollary 5.1, Proposition

5.1.

Theorem 5.1. *The image of the program p (optimization series-parallel system) via the submersion R of components*

$$R_i = 1 - \prod_{j=1}^{k_i} (1 - r_{ij})^{x_{ij}}, \ i = 1, 2, ..., n,$$

is the program P (optimization series system).

Remark 5.1. *A submersion locally looks like a projection* $\mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$*, while an immersion locally looks like an inclusion* $\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$*.*

Corollary 5.1. *To each optimal program P there corresponds an infinity of optimal programs p.*

A very nice original idea is dualities between reliability systems and electric circuits and the main result is

Proposition 5.1 *The total reliability of a series/parallel system is a copy of the first/second Ohm law for an electric circuit, via logarithmic scale.*

Example

Let us find *"reliability batteries" of minimum intensity* as counterpart of *electrical batteries of maximum intensity.*

Genuine example of batteries with maximum intensity

Let us give N identical electric cells, each with emf E and inner resistance r. Let us denote by R the exterior resistance. Tying in series n cells and then in parallel the groups so obtained, we form a battery. Determine n so that the battery to supply a current of maximum intensity.

Solution The current intensity given by such battery is

$$I = \frac{NnE}{NR + n^2 r}.$$

We think $n \to I(n)$ as a function on $(0, \infty)$ and we remark that

$$\max I = \frac{1}{\min \frac{1}{I}}.$$

On the other hand,

$$\min_{n>0} \frac{1}{I} = \min_{n>0} \left(\frac{R}{E} n^{-1} + \frac{r}{NE} n \right).$$

This is a posynomial geometric program with the solution $n = \sqrt{N\frac{R}{r}}$ and $I_{\max} = \frac{E}{2}\sqrt{\frac{N}{Rr}}$. If n is supposed to be a natural number, then the geometric program should be solved in steps.

Dictionary To pass to the reliability domain, we use

$$\mathcal{E} = e^E,\ \mathcal{I} = e^I,\ r \to -\ln \mathcal{R},\ R \to -\ln \mathcal{R}_1.$$

It follows

$$-\ln \mathcal{I} = \frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}} = log_{\mathcal{R}_1^N \mathcal{R}^{n^2}} \mathcal{E}^{Nn},$$

i.e.,

$$\mathcal{I} = exp\left(-\frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}}\right).$$

Dual example of reliability batteries with minimum intensity

Consider a reliability subsystem with N components, with the same reliability r. The components are grouped many n in series, and the series in parallel. To the total system we attach a subsystem, consisting of a single element with reliability R, connected in series.

The total reliability is

$$\mathcal{R}_s = [1 - (1 - r^n)^{N/n}]R.$$

We use a sequence of arrows based on previous commutative diagrams,

$$\mathcal{R}_s = [1 - (1 - r^n)^{N/n}]R \ \to\ -\ln[1 - (1 - r^n)^{N/n}] - \ln R$$
$$\to\ -\ln \frac{1}{(1 - r^n)^{N/n}} - \ln R \ \to\ \frac{n}{N}\ln(1 - r^n) - \ln R \ \to\ \frac{n}{N}\ln\frac{1}{r^n} - \ln R$$
$$\to\ -\frac{n^2}{N}\ln r - \ln R \ \to\ \frac{n^2}{N}r + R = \frac{nE}{I}.$$

The analog "reliability emf" of the battery is $\mathcal{E}$. To close the loop, we consider that the analog "reliability intensity" is

$$\mathcal{I} = c(r;n)(1 - \mathcal{R}_s) = exp\left(-\frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}}\right).$$

The function $n \to \mathcal{I}(n)$ has minimum for

$$n = \sqrt{N\ \frac{\ln \mathcal{R}_1}{\ln \mathcal{R}}}.$$

Generated algebraic structures

The commutative monoid $([0,1], S) : aSb = ab$ is isomorphic to the commutative monoid $([0,1], P) : aPb = 1-(1-a)(1-b)$ via an isomorphism f. For example, $f(x) = 1-x$ (involution). The general isomorphism is of the form $f(x) = h(1-h^{-1}(x))$, where $h : [0,1] \to [0,1]$ is an arbitrary bijection.

The commutative monoid $([0,+\infty], s) : AsB = A+B$ is isomorphic to the commutative monoid $([0,+\infty], p) : ApB = \frac{1}{\frac{1}{A}+\frac{1}{B}}$ via an isomorphism g. As example $g(y) = \frac{1}{y}$ (involution). Generally, the isomorphism is of the form $g(y) = \ell(1/\ell^{-1}(y))$, where $\ell : [0,+\infty] \to [0,+\infty]$ is an arbitrary bijection.

The commutative monoid $([0,1], S)$ is isomorphic to the commutative monoid $([0,+\infty], s)$ by the isomorphism $\varphi(x) = -\ln x$. Then $([0,1], P)$ is isomorphic to $([0,+\infty], p)$ by the same function $\varphi(x) = -\ln x$ if and only if $g(-\ln f^{-1}(x)) = -\ln x$.

Open question

Does exist a bijective and decreasing morphism $\varphi : [0,1] \to [0,+\infty]$ such that
$\varphi(aSb) = \varphi(a)s\varphi(b)$, i.e., $\varphi(ab) = \varphi(a) + \varphi(b)$;
$\varphi(aPb) = \varphi(a)p\varphi(b)$, i.e., $\varphi(a+b-ab) = \frac{\varphi(a)\varphi(b)}{\varphi(a)+\varphi(b)}$?

The original results are published in [1] Abed, S. A., Udrişte, C., & Ţevy, I. (2017). *Optimal reliability allocation for redundancy series - parallel systems.* Eur. J. Pure Appl. Math., 10(4), 873-885.

Chapter 6, titled **Optimal control on reliability problems**, introduces the optimal control to study *optimal value of mean time to failure functional* of a system. Here the basic problem is an optimal control with Bolza payoff and whose evolution ODE is a reliability flow. The original results include: (i) a list of payoffs with reliability sense, (ii) optimal value of mean time to failure functional constrained by controlled queueing system, (iii) optimal mean time to failure constrained by controlled Markov system, and (iv) optimal value of mean time to failure functional associated to parallel systems.

The original results in this Chapter are: Lemma 6.1, Theorem 6.1, Proposition 6.1.

Remark 6.1 *Let $R(p_0, ..., p_N)$ be the associated reliability polynomial and $R(t) = R(p_0(t), ..., p_N(t))$ be its pullback along the curve*

$$p(t) = (p_0(t), ..., p_N(t)),\ t \in [0, \infty),$$

whose image lives in $[0,1]^n$. Suppose the curve is such that the integral $\int_0^\infty R(t)dt$ is convergent. Otherwise, we can use a decreasing weight to ensure convergence.

Theorem 6.1 *If* $\Lambda(s)$ *has constant sign on the interval* $[0, t]$*, then the optimal states are*

$$p(t) = e^{At}c + (rI + A)^{-1}(e^{At} - e^{-rI})\ sgn\,\Lambda(s).$$

If s_0 *is a switch point on* $[0, t]$*, and* $\epsilon_0 = sgn\,\Lambda(s)$*, for* $s \leq s_0$*, and* $\epsilon_1 = sgn\,\Lambda(s)$*, for* $s \geq s_0$*, then the optimal states are*

$$p(t) = e^{At}c - e^{At}(rI + A)^{-1}\left[(e^{(rI+A)s_0} - I)\epsilon_0 + (e^{-(rI+A)t} - e^{-(rI+A)s_0})\epsilon_1\right].$$

Proposition 6.1 *Generally, the solution of the system* $\dot{p} = \mathcal{A}p$ *is* $p(t) = e^{\mathcal{A}t}c,\ t \in \mathbb{R}$.

The original results will be published in [142] Udrişte, C., Ţevy, I., & Abed, S. A. (2017). *Optimal control on reliability problems.* Far East Journal of Dynamical Systems, accepted for publication.

Future Work

Reliability has become an even greater concern in recent years because high-tech industrial processes with increasing levels of sophistication comprise most engineering systems today. Most papers in this area are devoted to: (i) multi-state system optimization, (ii) percentile life employed as a system performance measure, (iii) active and cold-standby redundancy, (iv) fault-tolerance mechanism, (v) optimization techniques (vi) bi-objective optimization. That is why, we shall study further the reliability Pareto optimization, starting from the model of bi-objective optimization with two objectives: the availability of the system and the total cost of the system.

Reliability is the probability that an item will perform a required function without failure under stated conditions for a stated period of time. Both efficiency and reliability are the most important factors for planning a project. The reduced efficiency and reliability of the global system is a direct consequence of the disorder caused as a result of the local active resources and the difficulty in managing these resources. Thus, the resulting resource allocation problem is also an entropy-optimization problem: how many resources should be allocated to a project in order to minimize average resource entropy, subject to limited cost budget within the examined time-period? The usual measures of system performance are basically of four kinds: reliability, availability, mean time-to-failure and percentile life. Due to the increasing complexity of practical engineering systems and the critical importance of reliability in these complex systems, our Thesis adds new functional objectives constrained by (deterministic or stochastic) ODE, PDE, formulating and solving optimal control problems. These problems still seem to be a very fruitful area for future research.The research dealing with the understanding and application of reliability at the nano level has also demonstrated its attraction and vitality in recent years. Optimal system design that considers reliability within the uniqueness of nano-systems has seldom been reported in the literature. It deserves a lot more attention in the future. In addition, uncertainty and component dependency will be critical areas to consider in future research on optimal reliability design. According Chapter 2, future work concerns: (i) nonlinear optimization problems formulated for multitime reliability polynomials; (ii) optimal control problems regarding the reliability and its optimal allocation. With regard to the Chapter 3, the advantage of our models is that the used mathematical technology can be applied to any system with high complexities. In contrast Chapter 4, the suggested model would be able to be applied to system design with a reliability goal with resource constraints for large scale reliability optimization problems. But a very difficult problem is to find a true interpretation of the coefficients and exponents used in a given posynomial. Regard to Chapter 5, the fundamental characteristic of our techniques is that similar techniques can be applied for simple and complex reliability systems. With regard to Chapter 6, future work concerns: (i) final refinements of our developed results, (ii) derive new functionals and new controlled ODEs, (iii) study the feasibility of imposing state constraints on the HIV model, (iv) optimal control problems regarding the reliability and its optimal allocation, (v) solve optimal control reliability problems with isoperimetric constraints (see (vii) in Section 1). Applications will be given in the area of robot control, control of chemical batch reactions and drug design of modeling HIV.

Author's published papers

- Udrişte, C., Abed, S. A., Rasheed, A. S. (2016). *Optimal reliability allocation.* American Review of Mathematics and Statistics, DOI: 10.15640/arms.v4n2a9, 4(2), 82-91.

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